Optimal Supply of Government Debt - Collateralization and Overaccumulation of Capital

Yong Hoon Cho*

Abstract

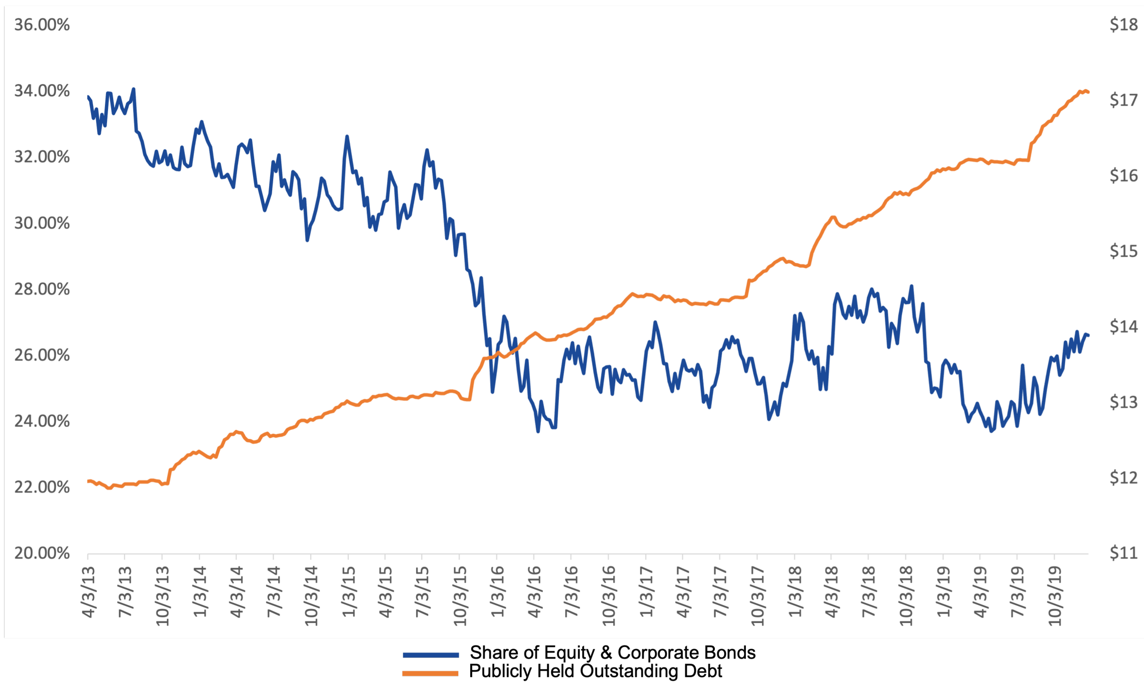
When capital serves as a substitute for government bonds in the collateral market, a high collateral premium results in an over-accumulation of capital as the firm demands more to relax the collateral requirement. Increasing the bond supply optimally crowds-out capital through two channels. First, it reduces the demand for capital by relaxing the collateral requirement. Second, because tax is distorting, a larger volume of debt level increases debt service and tax on capital. The optimal supply of government debt achieves efficient capital allocation. When the production uncertainty increases, a larger volume of government debt is needed to achieve the efficient allocation as agents demand larger amounts of collateral.

1 Introduction

The United States has faced several debt ceiling crises, the most recent of which was resolved in June 2023. The concept of a debt limit was introduced in 1917, at a time when the nation was expanding economically and engaging in war. To avoid the need for Congress to approve every single issuance of public debt, a pre-set limit was established, within which the Treasury had the discretion to issue debt. Since then, the role of government debts in the market has evolved, just like the debt ceiling has. Government debts not only act as a secure savings instrument but also frequently serve as collateral in the market. For instance, in 2002, the daily average volume

*Department of Economics, University of Virginia, Email:yc6tu@virginia.edu
I am grateful to my advisor Eric Leeper for inspiration and guidance. I thank Eric Young and Zach Bethune for helpful comments. I gratefully acknowledge the support of the Bankard Fund for Political Economy. All errors are my own.

of treasuries accepted by the New York Fed in the Repo (repurchase agreement) and reverse repo transactions was \$3.09 billion. By contrast, in 2022, the daily average had soared to \$991.91 billion. The rapid growth in the use of treasuries in the collateral market prompted the Federal Reserve to establish the Reverse Repo Facility in 2014 and the Repo Facility in 2020. These initiatives demonstrate the Fed's commitment to direct involvement in the repo market. Simultaneously, as the utilization of treasuries in the financial collateral market has increased, the use of private sector-issued financial instruments has declined, as shown in the figure below.



The figure shows the publicly held outstanding debt from 2013 to 2019 and the share of privately issued financial instruments in the collateral market. As the bond supply increased, these private sector instruments were used less in the collateral market. This trend raises a question about whether there is a correlation between the circulating volume of government debt, due to treasuries' liquidity service, and the utilization and thus the valuation of privately issued financial instruments. Krishnamurthy and Vissing-Jorgensen (2010), along with Gorton, Lewellen, and Metrick (2014), provide empirical evidence suggesting that private assets can substitute for treasuries as collateral when the supply of the latter is low.

This paper explores how the collateralization of government bonds and capital affects the firm's capital decision rule, and thereby impact welfare. The model features government debt serving both as safe saving device and as collateral, and capital acting both as a production input and as collateral. Expanding the supply of government debt benefits the economy by easing the collateral constraint and reducing the future risk of such constraints becoming binding. However, this necessitates a higher distorting tax rate to finance the debt service. At the optimal level, the benefits of increased debt supply are precisely counterbalanced by the necessity for higher taxes. A higher variance in productivity shock necessitates a larger volume of government debt to achieve an optimal steady state level of welfare. Furthermore, as the magnitude of precautionary saving escalates, so too does the optimal level of government debt. Certain aspects of this model echo those found in Aiyagari and McGrattan's (1998) work. Their model also proposes government debt that can improve welfare by easing constraints. However, the model presented in this paper diverges from Aiyagari and McGrattan's in the aspect of capital substitutability in the collateral market. This distinction allows the supply of government debt in our model to enable households to decrease the volume of capital held for collateral purposes. Consequently, government debt can further improve welfare by reducing the variance in consumption. The paper contributes to literature by providing a micro-foundation on how bond supply has a welfare impact and how it can stabilize the financial market through adjusting to output volatility.

This paper introduces a stylized model that highlights the inefficiencies that can arise when private assets provide liquidity services, and the role that fiscal policy can play in addressing these inefficiencies. In real life, when a financial firm takes out a collateral loan, the objective is to secure liquidity without surrendering asset ownership. This need for liquidity may stem from institutional requirements, or it may be due to the desire to make short-term investments. By obtaining liquidity in this manner, the firm can make additional investments without having to alter its existing asset holdings. This paper introduces a stylized model that explores the behavior of financial institutions in the collateral loan market. The firm in the model is representative of financial institutions and small businesses together. The financial institution posts its wealth as collateral to make short-term investment to the small businesses. To meet the payoff to the investment, small businesses hire labor with the investment. Here, the production with labor is comparable to making short-term investments, while capital is viewed as a longer-term investment, as it takes longer to put into production and lasts longer if it does not fully depreciate each period. Thus, the stylized firm in the model represents both the financial and production sectors. An implicit bank, not formally included in the model, facilitates loans from both sides: when the firm posts collateral for a short-term loan, the implicit bank takes the collateral and lends to the firm, which invests in small businesses to produce intermediate goods (represented by the household's labor provision), which the firm then uses to produce the final good and repay the loan to the bank. As we assume that the household owns the small business, financial institution, bank, and production firm, we can simplify this to a stylized model featuring only the household and the firm, with the firm facing financial constraints that the implicit bank facilitates.

The mechanisms of collateralization for government bonds and capital are as follows: In the model, the firm produce using capital and labor, with the firm owning the capital and households providing the labor. Because the firm lacks commitment, they pay households in advance for labor provision, requiring them to take out loans as production has not yet occurred. To make these loans, the firm must post collateral, which can include both their own capital and government bond holdings. This results in both capital and bonds carrying a collateral premium. Due to the fact that capital is more than just an input for production, the firm desire to accumulate a larger volume of capital than is efficient in each period. When production volatility increases, the firm's overaccumulation exacerbates, as it requires more collateral due to the higher volatility, and thus value capital more, leading to a larger overaccumulation of capital. Under the resource constraint, this crowds out households' consumption, and as a result of the overaccumulation of capital, output volatility increases. An increase in the bond supply will alleviate the need for collateral for the firm, reducing the overaccumulation of capital. However, this also leads to an increase in debt service, which the government finances through taxation, resulting in higher tax rates that distort capital accumulation. I define the optimal supply of government bonds as the bond supply that achieves efficient capital decision - the level of capital holding in the equilibrium without financial friction, and show that there exists an optimal level of bond supply that achieves efficient allocation. With an analytical example of the framework, I fully characterize the optimal supply of government bonds, and show that an increase in production uncertainty requires a larger bond supply to achieve efficient allocation. In a more general setting, through stochastic steady-state inference, I show how changes in bond supply affects steady-state outcomes by propagating through variance and covariance terms of endogenous variables. Finally, I numerically compute the policy functions and find the optimal bond supply assuming the economy was at the steady-state with no carried over debt. Here, I show that the optimal bond supply needs to increase as the production uncertainty (the standard deviation of the productivity shock) increases.

The results have a mixed flavor of Friedman (1969) and Diamond (1965). It is reminiscent of Friedman in the sense that it argues that closing the spread between private and government-issued assets is desirable, and Diamond in that government debt optimally crowds out overaccumulated capital. However, in contrast to Friedman, this carries an interesting trade-off between closing the spread and increasing distorting taxes. Compared to Diamond, the mechanism of overaccumulation and the way in which government bonds optimally crowd out capital is different, as in Diamond, tax was non-distortionary and agents overaccumulated capital to smooth consumption, while in this paper, the financial friction is the main source of the overaccumulation. Lastly, as mentioned, the model and its result shares certain aspect with that is introduced in Aiyagari and McGrattan (1998). Two works are indeed closely related to each other and addresses same issue. However, my work features capital substitutability that brings a whole new dynamics of the trade-off that bond supply brings. While in Aiyagari and McGrattan's, labor provision can substitute government bonds for its role as collateral, in my work, capital does so. This creates a new interesting relationship between real interest rate, marginal product of capital and the price of government bonds, which the government internalizes as it makes policy decisions. Moreover, this paper further shows the relationship of the production uncertainty and the optimal supply of government bonds.

The collateralization of government bonds had long been discussed in the literature. Chari and Kehoe (1999) argued that it is optimal to eliminate the liquidity premium on government debt when it provides liquidity services, similar to the Friedman rule. However, Calvo (1978), Woodford (1990), and Sims (2022) showed that when the tax is distorting, it's not optimal to eliminate the liquidity premium. Gorton, Lewellen and Metrick (2012) and Krishnamurthy and Vissing-Jorgensen (2010) showed the empirical evidences that the safeness of treasuries award them liquidity premium, and that when the supply of treasuries is low, private assets substitute them as collateral. Few studies have focused on the optimal policy when private assets substitute government bonds as collateral. Gorton and Ordonez (2014, 2020) provided a framework for the substitution of government bonds with private assets in the collateral market and concluded that it's not optimal to eliminate the collateral premium due to information asymmetry driving financial volatility. Berentensen and Waller (2017) discusses the liquidity premium on government debt and how the premium interacts with the optimal policy regarding the price level interaction. This paper contributes by defining and characterizing the optimal supply of collateralizable government bonds when private assets can substitute government bonds. Additionally, it shows how the optimal bond supply should respond to changes in production uncertainty. An increase in production uncertainty leads to a higher collateral premium and inefficient accumulation of capital, which requires an increase in bond supply to reduce the collateral premium and bring capital to the efficient level. This allows policy makers to determine how many treasuries to absorb or release to the collateral market.

In the next section, I lay out the model framework. Section 3 characterizes the equilibrium results. I define the optimal fiscal policy and prove the existence of the optimal supply of government bonds. In Section 4, I introduce an analytical example where I fully solve for the optimal bond supply. I show that with increasing productivity uncertainty, collateral premium increases, and the optimal bond supply increases.

2 The Model

The model introduced in this section is a stylized model that captures the macroeconomic behaviors of government debt collateralization. Government bonds and privately issued assets are both used as collateral in the financial market. Aside from its role as a collateral and safe saving device, the supply of government bonds affects the government's solvency conditions and tax policies. On the other hand, provision of private assets is linked with investment decisions. In this model, I propose an environment where government bonds and private investment are required for the collateral purpose, creating an interesting trade-off between having to carry larger amount of government bonds or private asset. This environment captures the macroeconomic climate of today in the aspect that financial instruments substitute each other whilst serving more than one purpose.

As discussed previously, the model setup is not to be taken literally as its purpose is to capture stylized facts in a concise way. The firm in the model represents financial sector and small businesses. The firm's employment of labor represents financial institutions' short-term investments to small business, and capital accumulation represents the issuance of private assets. To elaborate, when a financial firm makes a short-term investment to a small business, the small business uses the investment to employ labor for production to meet the investment payoff. Produced intermediary goods from the small businesses serve as an input for production for the economy. On the other hand, privately issued assets promote investment to larger firms, where we assume such investment are realized in the economy as capital investment. In aggregate, such interaction can be captured in a simple model with a firm and an household where the firm is required to post collateral in order to employ labor supplied by the household.

Although this model does not capture all the details of financial investments and other macroeconomic activity, it captures macro-behavior of private substitution of safe assets in the collateral market and its complications in macroeconomic policy decisions and investment decisions in a concise way. Using this model, we can understand how the bond supply affects policy variables and investment decisions.

2.1 Environment

The model operates in a discrete, infinite time horizon and features three types of agents: the household, the firm, and the government. There are continuum of homogeneous households in a unit mass. Households own the firm, with the firm producing consumption goods using capital and labor. Due to a lack of commitment from the firm to pay wages, households require the firm to pay wages before labor is provided. Because the production has not materialized, the firm cannot pay the wage directly to workers. Workers demand collateral from the the firm, which can be its capital or government bond holdings. When production takes place, the firm pays workers what it owes, settling the loan. In the case of default, the workers seize collateral equal to the value of their wage. To eliminate the chance of default, workers only take the collateral of the same value as the owed wage. The government issues bonds, pays interest on them, and collects taxes. After production and payments are settled, the firm invests in capital and households consume.

2.2 The Household

A household maximizes expected lifetime utility by solving following problem:

$$E_t(\sum_{j=0}^{\infty}\beta^j u(c_{t+j})) \tag{1}$$

subject to

$$c_{t+j}+b^h_{t+j+1}=w_{t+j}n_{t+j}+R_{t+j}b^h_{t+j}+\pi_{t+j}+z_{t+j} \tag{2}$$

where b^h_{t+1} is the household's government bonds holding that is issued and sold at t that pays interest R_{t+1} at $t+1$. π represents profits made by the firm, which is transferred to the household. z_t is the governmental transfer paid to the household, where when $z_t<0$, it becomes a lump-sum tax. We assume the household supplies one unit of labor inelastically ($n_t=1$). As each household owns a firm, they receive profit transfers, suggesting the firm discounts the future at households' stochastic discount rate.

2.3 The Firm

The firm solves following problem to maximize expected profit:

$$E_t(\sum_{j=0}^{\infty}m_{t,t+j}\Pi_{t+j}) \tag{3}$$

where

$$\Pi_{t+j}=(1-\tau_{t+j})A_{t+j}f(K_{t+j},N_{t+j})+(1-\delta)K_{t+j}-K_{t+j+1}+R_{t+j}B^f_{t+j}-B^f_{t+j+1}-W_{t+j}N_{t+j} \tag{4}$$

subject to

$$W_{t+j}N_{t+j}\leq(1-\delta)K_{t+j}+R_{t+j}B^f_{t+j}. \tag{5}$$

The firm discounts the future at households' stochastic discount rate, $m_{t,t+1}\equiv\beta\frac{u'(C_{t+1})}{u'(C_t)}$.The firm produces using the production function $f(K_t,N_t)$, where $f(\cdot)$ exhibits the normal properties of production functions. Capital depreciates at a rate of δ after each period of production usage. The firm's bond purchase, B^f_{t+1}, matures at $t+1$. The random productivity parameter A_t follows an identical, independent process, which is revealed at the start of each period, but is unknown at the time of the capital accumulation decision for K_t. Constraint (5) represents the firm's collateral constraint: when it hires workers, instead of paying the wage in advance, it posts collateral using capital and government bonds that mature at the end of the period. The workers accept depreciated capital as collateral, since by the time the firm repays the loan, the carried-over capital from today will have been used in production and will have depreciated. Since the firm has no incentive to borrow more than the wage, it only borrows the wage W_tN_t. This does not affect the workers' decision rule, as they simply wants to ensure that the firm has sufficient collateral to pay for the wage.

2.4 The Government

The government issues bonds, pays interest, collects tax, and pays transfers to households. Tax collected by the government is in the form of output tax. The budget constraint of the government is given by

$$B_{t+1} + T_t = R_t B_t + Z_t \tag{6}$$

where

$$T_t = \tau_t A_t f(K_t, N_t). \tag{7}$$

Z_t becomes lump-sum tax when $Z_t < 0$. When the government sets $\tau_t = 0$, it finances the debt service through the issuance of new debt and lump-sum tax. When $Z_t > 0$ and $\tau_t > 0$, the government finances the debt service and transfer through the distorting tax. As we move on, we discuss the scenarios where the government collects tax only though only distorting or lump-sum tax.

2.5 Market Clearing Conditions

Due to the unit masses of homogeneous agents, a household's decision variables c, b^h, n and transfer from the firm π aggregate and clear the market as follows:

$$c_t = C_t, \quad n_t = N_t = 1, \quad b^h_{t+1} = B^h_{t+1} \quad \pi_t = \Pi_t. \tag{8}$$

The resource constraint is given by

$$C_t + K_{t+1} = (1-\delta)K_t + A_t f(K_t, N_t) \tag{9}$$

and the asset market clears:

$$B^h_{t+1} + B^f_{t+1} = B_{t+1}. \tag{10}$$

The government's transfer aggregates: $z_t = Z_t$.

3 Equilibrium Conditions

In this section, I discuss how the bond supply affects the capital decision rule in the equilibrium. I first introduce the equilibrium results in general setting, and then introduce how the bond supply affects capital decision when tax is lump-sum or distorting. When tax is lump-sum, an increase in the bond supply reduces the collateral premium, which reduces capital accumulation. When tax is distorting, an increase in the bond supply reduces capital accumulation through two channels - first by increasing the distorting tax rate, and second by reducing the collateral premium.

3.1 Efficient Equilibrium Condition

We define the efficient equilibrium condition as an equilibrium condition with no financial frictions or tax distortions, where there is no collateral constraint or distorting tax. This scenario results in the government having no role in the economy, yielding a condition for the efficient allocation of capital, represented by K^*_{t+1}, where it satisfies

$$f_k(K^*_{t+1}) = \frac{1 - E_t(m_{t,t+1}(1-\delta))}{E_t(m_{t,t+1}A_{t+1})}. \tag{11}$$

The derivative of $f(\cdot)$ with respect to k is denoted as $f_k(\cdot)$. As per the standard properties of production functions, $f_k(\cdot)$ is a decreasing function in its argument. The efficient level of capital is influenced by the stochastic discount rate $E_t(m_{t,t+1})$, the depreciation rate δ, expected productivity A_{t+1}, and the risk discount associated with the productivity, $cov(m_{t,t+1}, A_{t+1})$. Thus, K^*_{t+1} increases with $E_t(m_{t,t+1})$, expected value of A_{t+1}, and the risk discount $cov(m_{t,t+1}, A_{t+1})$. Note that the risk discount, being a negative number, varies with the distribution of A_{t+1}, and a higher variance leads to a larger risk discount. An increase in production uncertainty, or the variance of A_{t+1}, results in an increase in both $E_t(m_{t,t+1})$ and $E_t(m_{t,t+1}A_{t+1})$, as agents increase their precautionary saving due to greater uncertainty of future income stream. Equilibrium level of consumption will be given by the resource constraint and the capital allocation decision. Note that this subsection is introduced to remind readers what a typical first-best outcome would be had there been no financial constraint or tax distortion.

In the following section, we characterize the competitive equilibrium with financial friction to compare with the efficient allocation result. As it will be shown, the efficient allocation result will serve as the benchmark to gauge the efficiency of the competitive equilibrium under financial constraint and governmental intervention.

3.2 Competitive Equilibrium with Financial Friction

The competitive equilibrium condition for the household is given by

$$E_t(m_{t,t+1}) = \beta E_t(\frac{u'(C_{t+1})}{u'(C_t)}) = \frac{1}{R^h_{t+1}} \tag{12}$$

By the household's equilibrium condition, we get the stochastic discount rate $E_t(m_{t,t+1})$ and the interest on the government bonds that the household is willing to receive, R^h_{t+1}.

The firm's equilibrium conditions are given by

$$K_{t+1} = f_k^{-1}(\frac{1 - E_t(m_{t,t+1}(1-\delta)(1+\eta_{t+1}))}{E_t(m_{t,t+1}(1-\tau_{t+1})A_{t+1})}) \tag{13}$$

$$W_t = \frac{(1-\tau_t)A_t f_n(K_t)}{1+\eta_t} \tag{14}$$

$$\frac{1}{R^f_{t+1}} = E_t(m_{t,t+1}(1+\eta_{t+1})) \tag{15}$$

where η_t represents the Lagrange multiplier for the firm's constraint (5) and can be interpreted as the collateral premium on capital and government bonds. If the collateral constraint (5) is binding, we have $\eta_t > 0$, and if not binding, $\eta_t = 0$. This indicates the extent to which the constraint is

binding and how much agents want to relax it. K_{t+1} is determined by the stochastic discount rate, expected collateral premium, expected tax rate, and expected productivity. As $f_k^{-1}(\cdot)$ is a decreasing function, K_{t+1} increases with A_{t+1}, $E_t m_{t,t+1}$, and $1+\eta_{t+1}$. A rise in productivity prompts the firm to invest more in capital, a rise in collateral premium makes the firm hold more capital to relax the constraint, and a rise in the stochastic discount rate prompts the firm to invest more in capital as the real interest rate of the household increases. W_t is the wage paid by the firm to the household for labor provision. Note that this also relies on the realized value of η_t: when the collateral constraint is binding, it reduces the wage that the household receives. To avoid this event, the household demands the firm to accumulate larger volume of collateral, and this force adds to the model dynamics which creates the inefficiency. R^f represents the interest on government bonds the firm is willing to receive. We can write this as $E_t(m_{t,t+1}) + E_t(m_{t,t+1}\eta_{t+1})$, where we can express the first term as the fundamental value of the government bond and the second term as the collateral premium of the government bond. Note that in an ideal scenario where the collateral constraint does not bind at all, the government bond valuation would be just the fundamental value of it.
Comparing (12) and (15), we see that the interest that households are willing to receive and the firm is willing to receive are different. Because the firm is willing to accept lower rate since $E_t(1+\eta_{t+1}) \geq 1$, we arrive at conclusion that

$$\frac{1}{R_{t+1}} = E_t(m_{t,t+1}(1+\eta_{t+1})). \tag{16}$$

If the probability distribution of A_{t+1} has an infinite support, we would have $E_t(1+\eta_{t+1}) > 1$ because there is always a non-zero probability that the collateral constraint will bind. I specify why this is the case with ***Observation 1*** in the following section.

3.2.1 Collateral Premium

Given that $1+\eta_t$ represents the collateral premium affecting the firm at the start of period t, we can examine its impact on the equilibrium conditions. By separating the fundamental value of government bonds $E_t(m_{t,t+1})$ and the collateral value $E_t(m_{t,t+1}\eta_{t+1})$ in equation (16), we conclude that the collateral premium on government bonds B_{t+1} sold at t is $E_t(m_{t,t+1}\eta_{t+1})$. This can also be applied to the capital decision rule (13), where the collateral premium on capital is $E_t(m_{t,t+1}(1-$

$\delta)\eta_{t+1})$. This is discounted by $(1-\delta)$ because capital depreciation occurs when capital enters the collateral constraint (5). Given the significant impact of the collateral value $E_t(1+\eta_{t+1})$ on the equilibrium level, its value must be formally defined and analyzed. The collateral premium $1+\eta_t$ is given by

$$1+\eta_t = \begin{cases} 1 & \text{if} \quad (1-\tau_t)A_t f_n(K_t) \leq (1-\delta)K_t + R_t B_t^f \\ \frac{(1-\tau_t)A_t f_n(K_t)}{(1-\delta)K_t + R_t B_t^f} & \text{if} \quad (1-\tau_t)A_t f_n(K_t) > (1-\delta)K_t + R_t B_t^f \end{cases}. \tag{17}$$

The value of $1+\eta_t$ is determined by the binding status of constraint (5), where the upper case of (17) indicates the condition that the collateral constraint would not bind, and the lower case of (17) indicates the condition that the constraint would bind. When the collateral constraint (5) does not bind, $\eta_t = 0$, so $1+\eta_t = 1$, but when it binds, we have $1+\eta_t = \frac{(1-\tau_t)A_t f_n(K_t)}{(1-\delta)K_t+R_t B_t^f}$. $E_t(1+\eta_{t+1})$, the expected collateral premium of government bonds sold today and capital chosen today, can be written loosely as

$$\begin{aligned} E_t(1+\eta_{t+1}) = {} & Pr((1-\tau_{t+1})A_{t+1}f_n(K_{t+1}) \leq (1-\delta)K_{t+1} + R_{t+1}B_{t+1}^f) \\ & + Pr((1-\tau_{t+1})A_{t+1}f_n(K_{t+1}) > (1-\delta)K_{t+1} + R_{t+1}B_{t+1}^f) \frac{(1-\tau_{t+1})A_{t+1}f_n(K_{t+1})}{(1-\delta)K_{t+1}+R_{t+1}B_{t+1}^f}. \end{aligned} \tag{18}$$

The RHS of (18) can be decomposed as the probability that the collateral constraint will not bind multiplied by the collateral premium under unbinding collateral constraint (i.e., 1), plus the probability that the collateral constraint will bind times the collateral premium when the collateral constraint (5) binds. While this expression is not formal, it allows one to see that as the collateral holding of the firm ($(1-\delta)K_{t+1} + R_{t+1}B_{t+1}^f$) increases, the collateral premium decreases, because the probability of the constraint binding decreases. Again, this is a very loose expression for illustrative purpose. This expression can be refined and written more accurately:

$$\begin{aligned} & E_t(1+\eta_{t+1}) = \\ & \int_0^{\frac{(1-\delta)K_{t+1}+R_{t+1}B_{t+1}^f}{(1-\tau_{t+1})f_n(K_{t+1})}} \Phi(A_{t+1})dA_{t+1} + \int_{\frac{(1-\delta)K_{t+1}+R_{t+1}B_{t+1}^f}{(1-\tau_{t+1})f_n(K_{t+1})}}^{\infty} \frac{(1-\tau_{t+1})A_{t+1}f_n(K_{t+1})}{(1-\delta)K_{t+1}+R_{t+1}B_{t+1}^f} \Phi(A_{t+1})dA_{t+1} \end{aligned} \tag{19}$$

where $\Phi(\cdot)$ is the probability distribution function of A_{t+1}. Note that if the probability distribution function has an infinite support in $\mathbb{R}^+$, the expected collateral premium is always positive, since there is a non-zero probability of collateral constraint binding, we have

$$E_t(1+\eta_{t+1}) > 1. \tag{20}$$

This is true no matter how large the volume of government bonds or capital stock a firm collects, as there always exists a probability that the collateral constraint will bind in the following period. In other words, it is impossible to diminish the expected collateral premium entirely - it will asymptotically converge to one. With this property, the interest on government debt that a household is willing to accept and a firm is willing to receive disagrees:

Observation 1. *When the probability distribution of the productivity parameter A_{t+1} has an infinite support, the interest on government bonds is given by $R_{t+1} = R^f_{t+1} < R^h_{t+1}$ by (12) and (15), where we acquire $B^h_{t+1} = 0$ and $B^f_{t+1} = B_{t+1}$.*

With ***Observation 1***, we can express the equilibrium capital decision and the collateral premium as a function of government bond supply B_{t+1}, instead of the firm's bond holding B^f_{t+1}. This means that when the government chooses the bond supply B_{t+1}, it will influence the firm's capital decision rule.

In the following section, we will explore how the bond supply affects the capital decision rule of the firm.

3.2.2 Over-accumulation of Capital

Now we compare the efficient allocation of capital to the competitive allocation. Recall that the efficient allocation is given by

$$f_k(K^*_{t+1}) = \frac{1 - E_t(m^*_{t,t+1}(1-\delta))}{E_t(m^*_{t,t+1}A_{t+1})} \tag{21}$$

and the competitive allocation is given by

$$f_k(K_{t+1}) = \frac{1 - E_t(m_{t,t+1}(1-\delta)(1+\eta_{t+1}))}{E_t(m_{t,t+1}(1-\tau_{t+1})A_{t+1})} \tag{22}$$

where terms with * notations indicate equilibrium results under the efficient allocation. Note that the capital decision rule under the competitive equilibrium is subject to the collateral premium and output tax rate contrast to the efficient allocation. Also note that if $K_{t+1} > K^*_{t+1}$, then we have $E_t(m^*_{t,t+1}) > E_t(m_{t,t+1})$: with higher capital investment, consumption at t falls, making the marginal utility at t increase.

To understand what a lack of bond supply does to the economy in this environment, it helps us to consider the case where the government does not supply any government bonds: $B_{t+1} = 0$. This would suggest that there is a high probability of the collateral constraint binding in the following period. This makes capital to carry higher collateral premium - $E_t(1+\eta_{t+1})$ increases. This makes the firm to accumulate larger volume of capital. As the firm accumulates larger volume of capital, by the resource constraint, the household consumption decreases, increasing the stochastic discount rate $E_t(m_{t,t+1})$. Feeding into the capital decision rule, this increases the capital demand of the firm. Overall, the firm would end up accumulating larger capital stock compared to the efficient allocation in the absence of the government bonds.

3.2.3 Lump-sum Tax Case

Suppose the government sets $\tau_t = 0 \forall t$. This changes the government's budget constraint as

$$B_{t+1} - Z_t = R_t B_t \tag{23}$$

where the government is financing the debt service through lump-sum tax and issuance of new debt, as $-Z_t$ is the lump-sum tax. This changes the firm's capital decision rule in the equilibrium as

$$f_k(K_{t+1}) = \frac{1 - E_t(m_{t,t+1}(1-\delta)(1+\eta_{t+1}))}{E_t(m_{t,t+1}A_{t+1})} \tag{24}$$

where the collateral premium $E_t(1+\eta_{t+1})$ is given by

$$E_t(1+\eta_{t+1}) =$$
$$\int_0^{\frac{(1-\delta)K_{t+1}+R_{t+1}B_{t+1}}{fn(K_{t+1})}} \Phi(A_{t+1})dA_{t+1} + \int_{\frac{(1-\delta)K_{t+1}+R_{t+1}B_{t+1}}{fn(K_{t+1}))}}^{\infty} \frac{A_{t+1}fn(K_{t+1})}{(1-\delta)K_{t+1}+R_{t+1}B_{t+1}} \Phi(A_{t+1})dA_{t+1}. \tag{25}$$

A noticeable difference of the lump-sum setup is that the distorting tax τ does not appear on the firm's decision rule - there is no distortionary effect. Further, from (25), we observe that the collateral premium $E_t(1+\eta_{t+1})$ is decreasing in B_{t+1}. An increase in B_{t+1} affects the household's discount rate $E_t(m_{t,t+1})$ by reducing the collateral premium and thus decreasing the capital accumulation, but not as a savings device on its own. As the government increases the bond supply, the collateral premium decreases, as the probability of collateral constraint binding decreases. When the tax is not distorting, it is easy to see that the optimal policy is to flood the market with bonds: as $B_{t+1} \to \infty$, the collateral premium $E_t(1+\eta_{t+1})$ converges to 1, where eventually K_{t+1} converges to K^*_{t+1}. This result echos with the Friedman rule and Chari and Kehoe's result, as the optimal policy is to close the spread between the government issued asset and privately issued asset. However, this result is rather uninteresting as it argues that an infinite fiscal expansion is costless and welfare improving to the economy. Because we know that tax is distorting in the real world, we discuss the case that the government is not taxing individuals with lump-sum tax ($Z_t \geq 0\ \forall t$) and finances the debt service through distorting tax in the following sections.

3.2.4 Distorting Tax Case

Now to explore how the bond supply affects the capital decision rule, suppose the government finances the debt through the distorting tax, $\tau_t > 0$. Furthermore, let $Z_t = 0\ \forall t$, so that we can examine the case where the debt service is financed sorely by the distorting tax. In this case, we see the effects of reduction in the collateral premium and distortion together as the bond supply increases. This suggests that increasing the bond supply brings an interesting trade-off: on the benefit side, the collateral premium decreases, and on the cost side, the distorting tax increases.

Bond Supply and Tax Decision Rule

The government's budget constraint (6) can be re-written as

$$\tau_t A_t f(K_t, N_t) = R_t B_t - B_{t+1}. \tag{26}$$

To nail down the expectation of the future tax rate, τ_{t+1}, when the government chooses the bond supply today (B_{t+1}), it establishes the expectation that there will be the same level of government bonds supplied in the next period - $E_t(B_{t+2}) = B_{t+1}$. The government commits to this tax rate. This then determines the tax rate as follows:

$$\tau_{t+1} = \frac{(R_{t+1} - 1)B_{t+1}}{E_t(A_{t+1})f(K_{t+1})} \tag{27}$$

where we see that τ_{t+1} is an increasing function of B_{t+1}. One can also note that depending on the realization of A_{t+1}, the future supply of government bonds may need to change. Even when the government changes the bond supply in the following period after the realization of A_{t+1}, the expectation regarding the bond supply does not change: that is, $E_{t+1}(B_{t+2}) = B_{t+1}$. Because the tax rate has to be within the range of $[0, 1)$, this rule sets an upper-bound for the bond supply that can be provided today:

$$B_{t+1} < \frac{E_t(A_{t+1})f(K_{t+1})}{R_{t+1} - 1}. \tag{28}$$

We also need to consider how the tax rate today is determined as the government decides the level of government bonds to supply. Given R_t, B_t, A_t, K_t and N_t, the tax rate τ_t is determined, where we require $\tau_t < 1$. This then gives the lower bound for the bond supply:

$$R_t B_t - A_t f(K_t) < B_{t+1}. \tag{29}$$

This provides a set of guideline to the government as it makes policy decision on how much bonds to supply.

Optimal Supply of Government Bonds

Because the firm accumulates capital beyond the efficient level without government bonds in the market, the government can increase the bond supply to reduce the capital accumulation as the firm's capital decision rule is a decreasing function of government bonds. We have seen that an increase in the government bond issues will 'crowd-out' capital slowly in the lump-sum tax case. When the debt is financed through distorting tax, the same effect will take place, and furthermore, the distortionay effect will be added. When the government commits to finance the debt service through the distorting tax, the increase in the government bond supply increases the expected tax rate. This crowds-out capital accumulation, which feeds into the capital decision rule in (22), negatively affecting the capital accumulation furthermore. Because the capital stock decreases, consumption at t increases, which in turn increasing the stochastic discount rate $E_t(m_{t,t+1})$, contributing to a further decrease of the capital.

The provision of government bonds can be welfare improving when agents overaccumulate capital due to its collateral value. As contrast to the lump-sum tax case where this was also true, the effect of increase in the bond supply is more pronounced in the distorting tax case. In the lump-sum scenario, we needed infinite bond supply to return the firm's capital decision to the efficient level. In the distorting case, however, we can achieve the efficient level of capital through a finite supply of government bonds. This is because distorting tax also crowds-out capital. In the case where the government has control over both lump-sum tax and distorting tax, the government will be able to achieve efficiency more flexibly, as they can costlessly finance the debt-service through lump-sum tax, while also be able to efficiently crowd-out capital with distorting tax.

In the following section, I show that there exists the optimal level of government bond supply through an analytical example.

4 Analytical Example

In this section, I fully characterize the optimal supply of government bonds that achieves the efficient allocation. Moreover, I explicitly show that an increase in the output shock volatility can increase the magnitude of over-accumulation of capital. This shows that the optimal supply of government bonds increases as the output volatility increases.

In this example, I use three simplifying assumptions to make the analysis tractable:

i) Linear Utility

I use linear utility for the tractability. Moreover, this allows us to focus on the collateral mechanism of government bonds more than the role as a safe asset. While this assumption preserves the result, it weakens the result somewhat - with risk aversion, agents will show more tendencies to increase capital accumulation when the collateral premium increases. However, the risk aversion can get in the way and hinder us from observing how the collateral usage of government bonds and capital's substitutability can result in the overaccumulation of capital. Another aspect that is lost by this assumption is the precautionary savings. With precautionary savings, the result would be more pronounced. However, because the result prevails even in the linear case, this assumption shows the robustness to the results. It also helps us solve for the optimal supply of government bonds by making model solution more tractable.

ii) Binomial Distribution

I choose binomial distribution for tractability. A binomial distribution, however, does not harm the result of the model environment if we choose a large enough support (that is, the distance between two results of the distribution is large enough). This is more so with this example as we choose linear utility because households do not demonstrate the need for precautionary saving or risk aversion - the shape of the distribution matters little to them. The variance of the distribution, however, will affect their decisions as will be shown. As we move along, we discuss what configuration of the binomial distribution can serve as the one analogous to a distribution that has an infinite support in the $\mathbb{R}^+$.

iii) Capital Tax

Output taxation creates distortion in capital accumulation, while also creating analytical swamp. For the model tractability again, I use taxation on capital holding, which also creates distortion (i.e., $T_t = \tau_t k_t$). Because tax on capital holding is less distortionary than output taxation, the result will be more pronounced with the output taxation, hence again shows the robustness of the results. However, I show that the result preserves with the capital holding taxation.

4.1 The Model

The household consumes at linear utility: $u(c_t) = c_t$ and the firm faces production technology $f(k,n) = ln(k) + ln(n)$, where the firm's profit function is given by

$$\Pi_t = A_t f(K_t, N_t) - W_t N_t - K_{t+1} + (1-\delta)K_t - Q_{t+1} B^f_{t+1} + B^f_t - \tau_t K_t. \tag{30}$$

Here, Q_{t+1} is the price of government bond sold at t, that matures at $t+1$ and yields one unit of consumption good. Unlike the general model setup provided in the previous part, we use this configuration in the analytical example because it simplifies the notation whilst preserving the equivalency in policy decisions and household behaviors.

The process for the productivity parameter A is given by

$$A_t = \zeta A_{t-1} + \varepsilon_t \tag{31}$$

for $\zeta \in (0,1)$ and ε_t takes following process:

$$\varepsilon_t = \begin{cases} \varepsilon^L & \text{w.p.} \quad \rho \\ \varepsilon^H & \text{w.p.} \quad 1-\rho. \end{cases} \tag{32}$$

such that

$$E_t(\varepsilon_{t+1}) = (1-\zeta)A \tag{33}$$

where A is a given long-run mean of the process for the productivity. We can re-write the process for A_t as

$$A_t = \begin{cases} A_t^L & \text{w.p.} \quad \rho \\ A_t^H & \text{w.p.} \quad 1-\rho \end{cases} \tag{34}$$

where $A_t^L = \zeta A_{t-1} + \varepsilon^L$ and $A_t^H = \zeta A_{t-1} + \varepsilon^H$. We assume $\varepsilon^L < \varepsilon^H$ where ε^H is a sufficiently large positive number, with $\rho > (1-\rho)$ such that (33) holds. We require this assumption to make the distribution analogous to a distribution with continuous and infinite support in $\mathbb{R}^+$. With sufficiently large A_t^H and ρ sufficiently close to 1, we can ensure that B_t^* such that $K_{t+1}(B_t^*) = K_{t+1}^*$ exists, which would be the case of the distribution with continuous and infinite support. We discuss what conditions A^L, A^H and ρ must satisfy in the following section.

Efficient Equilibrium Conditions

Because of the linear utility, the stochastic discount rate $E_t(m_{t,t+1})$ would be given by

$$E_t(m_{t,t+1}) = \beta E_t(\frac{u'(c_{t+1})}{u'(c_t)}) = \beta. \tag{35}$$

This yield the efficient capital allocation given by

$$K^*_{t+1} = \frac{\beta E_t(A_{t+1})}{1-\beta(1-\delta)}. \tag{36}$$

Competitive Equilibrium Conditions

In the competitive equilibrium, the equilibrium price of government bond is given by

$$Q_{t+1} = \beta E_t(1+\eta_{t+1}). \tag{37}$$

As shown in the previous parts of the paper, the firm pays higher price for government bonds than households, resulting in no purchase of government bonds by households. This makes the firm the sole holders of government bonds:

$$B^h_{t+1} = 0 \quad B^f_{t+1} = B_{t+1}. \tag{38}$$

Equilibrium wages are given by

$$w_t = \frac{A_t}{1+\eta_t} \tag{39}$$

where

$$1+\eta_t = \begin{cases} 1 & \text{if} \quad A_t \leq (1-\delta)K_t + B_t \\ \frac{A_t}{(1-\delta)K_t+B_t)} & \text{if} \quad A_t > (1-\delta)K_t + B_t. \end{cases} \tag{40}$$

The collateral premium on B_{t+1} depends on how we configure the parameter values. Recall that from the previous part the collateral premium was given by (18) where it was described as probability of collateral constraint not binding times 1 plus the probability of the collateral constraint binding times the bottom value of (40). To make the example analogous to the general case, we configure the parameter values such that we can write the collateral premium as

$$E_t(1+\eta_{t+1}) = \rho + (1-\rho)\frac{A^H_{t+1}}{(1-\delta)K_{t+1}+B_{t+1}}. \tag{41}$$

For this to hold true, we need sufficiently large ρ and A^H_{t+1} such that condition 64 to hold. This requirement is discussed in detail in Appendix 9.1.1. This allows us to write $E_t(1+\eta_{t+1})$ as given in (41).

The capital decision rule in the general equilibrium is given by

$$K_{t+1} = \frac{\beta E_t(A_{t+1})}{1+\beta E_t(\tau_{t+1}) - \beta(1-\delta)E_t(1+\eta_{t+1})}. \tag{42}$$

Note that K_{t+1} is decreasing in τ_{t+1}, the tax rate on capital, and $1+\eta_{t+1}$, the collateral premium. It is easy to see that to achieve the efficient allocation, $\tau_{t+1} \to 0$ and $\eta_{t+1} \to 0$ is desirable. As will be revealed along the way, we can express the capital decision rule as a quadratic function of the government bond supply from this.

The government's budget constraint is given by

$$\tau_t K_t + Q_{t+1}B_{t+1} = B_t + Z_t. \tag{43}$$

In this example, for illustrative purpose and simplicity, we assume that the government sets the transfer Z_t equal to the incoming revenue from bond sales: $Z_t = Q_{t+1}B_{t+1}$. This allows us to identify the tax decision rule as simple as

$$\tau_{t+1}K_{t+1} = B_{t+1}. \tag{44}$$

With this simplification, we can directly plug in τ_{t+1} to the capital decision rule to solve the equilibrium capital decision as a function of government bonds:

$$K_{t+1} = \frac{\beta E_t(A_{t+1})}{1+\beta \frac{B_{t+1}}{K_{t+1}} - \beta(1-\delta)(\rho + (1-\rho)\frac{A^H_{t+1}}{(1-\delta)K_{t+1}+B_{t+1}})}. \tag{45}$$

This can be further expanded as a quadratic equation for K_{t+1} in terms of parameter values and the bond supply (detailed in Appendix 9.1.2.).

Consider the case where the government does not supply any government bond: $B_{t+1} = 0$. Then, this would result in the capital volume

$$\bar{K}_{t+1} = \frac{\beta E_(A_{t+1}) + \beta(1-\rho)A^H_{t+1}}{1-\beta(1-\delta)\rho} \tag{46}$$

where we observe that $\bar{K}_{t+1} > K^*_{t+1}$. This means that in the absence of the government bonds, the competitive capital investment exceeds the efficient level of capital investment - in other word, the economy suffers inefficiency from overaccumulation. As it is observable from the model environment, this is because of capital's role as collateral. In the absence of the government bonds, capital is the sore collateral available in the economy, so the firm accumulates capital more than what it needs for production alone. With such overaccumulation, if the distortionary effect of increasing the bond supply is large enough, we can find B^*_t that achieves the efficient allocation: by choosing the bond supply $B_{t+1} = B^*_t$, the government can achieve $K_{t+1}(B^*_t) = K^*_{t+1}$. Algebraically, this can be done by equating (45) to (36) and solving for B_{t+1} such that the equation is satisfied.

4.2 Optimal Supply of Government Bonds

The optimal supply of government bonds would solve following:

$$K_{t+1}(B_{t+1}) = K^*_{t+1} = \frac{\beta E_t(A_{t+1})}{1-\beta(1-\delta)}. \tag{47}$$

The positive solution of the quadratic equation for K_{t+1} is given by

$$\begin{aligned} B^*_t = \frac{1}{2\beta}\Bigg[(\beta E_t(A_{t+1})+\xi\rho) - K^*_{t+1}(1+\xi) + \Big((K^*_{t+1}(1+\xi) - (\beta E_t(A_{t+1})+\xi\rho))^2 + \\ +4\beta(\xi(E_t(A_{t+1})+(1-\rho)A^H_{t+1}) - (1-\delta)(1-\xi\rho)K^*_{t+1})K^*_{t+1}\Big)^{\frac{1}{2}}\Bigg]. \end{aligned} \tag{48}$$

where $K^*_{t+1} = \frac{\beta E_t(A_{t+1})}{1-\beta(1-\delta)}$ and $\xi \equiv \beta(1-\delta)$. (48) characterizes the optimal supply of government bonds for this economy that achieves the efficient allocation $K_{t+1} = K^*_{t+1}$ (detailed derivation can be found in Appendix 9.1.3). Because of the complexity of the solution, it is not clear to see the characteristics of this solution. Plotting this gives us a better understanding of what the optimal supply of government bonds does.

Figure 1-1[1] shows the plot for K_{t+1} as a function of B_{t+1} in the showcased example above. Figure 1-2 on the other hand, is the case when the bond is financed only through lump-sum tax. Note that as the supply of government bonds increases, the capital level converges to the efficient

[1] $\delta = 0.3$, $\beta = 0.95$, A=10, $\rho = 0.91$

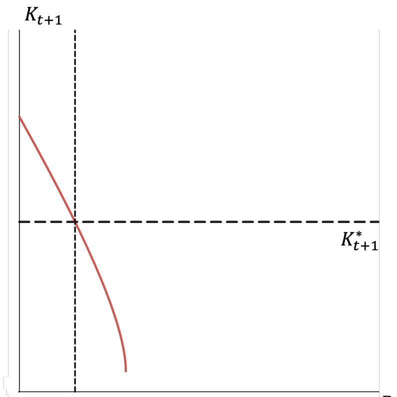


Figure 1-1

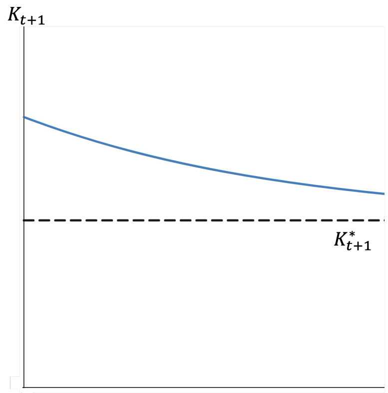


Figure 1-2

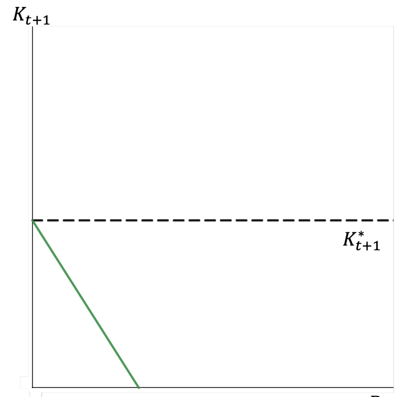


Figure 1-3

level. Because in this example, the distribution of A_t is binomial, it will eventually reach K_{t+1}^* as the level of B_{t+1} increases to a large magnitude. However, when the distribution of A_t has an infinite support as discussed in the previous part, it will converge asymptotically to K_{t+1}^* as B_{t+1} goes to infinity. Figure 1-3 is the case when the firm is not subject to the collateral constraint and the tax is distorting (capital tax). Note that the capital accumulation is at the efficient level when $B_{t+1} = 0$, where $\tau_t = 0$.

4.3 Output Volatility and Optimal Supply of Government Bonds

As the standard deviation of the productivity shock increases, the optimal supply of government bonds increases. This is because when the production volatility increases, the firm demands larger volume of capital for collateral purpose, making collateral premium on capital and government bond increases. This causes a larger over-accumulation of capital. To relax the collateral constraint and crowd out the over-accumulated capital, the government can increase the bond supply. At the optimal supply of government bonds and efficient capital allocation, the collateral premium is non-zero. The analytical derivation of the analysis is included in Appendix 9.1.3., where I show that the optimal supply of government bonds is an increasing function of the standard deviation.

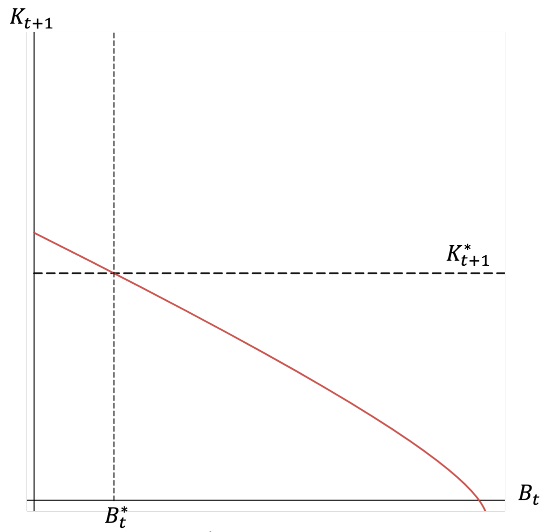


Figure 2-1

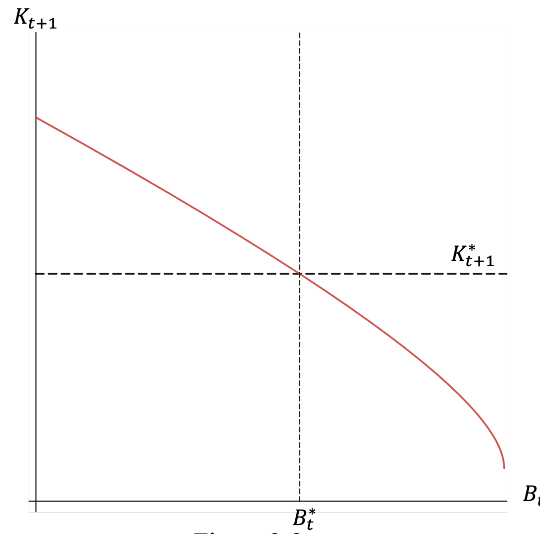


Figure 2-2

Figure 2-1[2] is the plot of K_{t+1} as a function of B_{t+1} when $\gamma^2 = 3$ and Figure 2-2[3] is when $\gamma^2 = 9$, where $A = 10$ and $\rho = 0.91$ for both. Note that the accumulation of capital when $B_{t+1} = 0$ increases significantly as the variance increases. The optimal supply of government debt, B^*_t, increases by a large magnitude as well as the response to the increase in the variance. The efficient level of capital K^*_{t+1} does not increase as a response to the increase in the variance as the household is risk neutral: households do not exhibit precautionary saving behavior.

5 Stochastic steady state

Because the mechanism of this model relies on the uncertainty of productivity, it is necessary to include the role of the uncertainty and risk-averse behavior of the household when investigating the model's steady state equilibrium. Thus, this section finds the stochastic steady states of the model following the method suggested by Coeurdacier, Rey and Winant (2011). Although the method falls behind in terms of precision of its approximation, it allows one to make analytical inference of the risk-averse behavior of agents. One should understand that although this steady-state analysis compare welfares of two different steady states, it overlooks the transitional paths between steady states, and should not be taken literally - it is merely a device to help our understanding. In the following section, I define the stochastic steady state and numerically solve for it.

[2] $A^L_{t+1} = 6.7, A^H_{t+1} = 43$
[3] $A^L_{t+1} = 0.1, A^h_{t+1} = 109$

Further, because we do not have to solve the model entirely analytically, it allows us to relax heavy simplifying assumptions used in the analytical example, and move to a more general setting. Here, I use following configurations to solve for the model:

$$U(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma} \tag{49}$$

and

$$F(K_t, N_t) = K_t^{\alpha} N_t^{1-\alpha}. \tag{50}$$

Let the process for the productivity A follow:

$$ln(A_t) = a_t \tag{51}$$

where

$$a_t = \zeta a_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim N(0, \sigma^2). \tag{52}$$

Here, $\zeta \in [0,1)$ is the auto-regressive persistence parameter and $\sigma > 0$ is the standard deviation. Derivations and inferences are detailed in Appendix 9.2. For this exercise, I use following parameter values: $\alpha = 0.33$, $\beta = 0.95$, $\delta = 0.1$, $\rho = 0$, and $A = 1$.

5.1 Steady state equilibrium conditions

Applying the steady state condition, we let $E_t(C_{t+1}) = C$, $K_{t+1} = K$, $\tau_{t+1} = \tau$ and $E_t(\eta_{t+1}) = \eta$, we first get the resource constraint in the steady state as

$$C = AK^{\alpha} - \delta K \tag{53}$$

and second order expansions of the equilibrium conditions simplify to:

$$\begin{aligned} &\beta((1-\tau)A\alpha K^{\alpha-1} + (1-\delta)(1-\eta)) + \frac{1}{2}\beta \frac{(1-\tau)A\alpha K^{\alpha-1} + (1-\delta)(1+\eta)}{C} Var_t(C_{t+1}) \\ &\qquad - \beta \frac{(1-\tau)\alpha K^{\alpha-1}}{C} Cov_t(C_{t+1}, A_{t+1}) - \beta \frac{(1-\delta)}{C} Cov_t(C_{t+1}, \eta_{t+1}) = 1 \end{aligned} \tag{54}$$

and

$$\frac{1}{R} = \beta(1+\eta) + \beta\gamma(\gamma+1)\frac{1+\eta}{2C^2}Var_t(C_{t+1}) - \frac{\beta\gamma}{C}Cov_t(C_{t+1},\eta_{t+1}). \tag{55}$$

Note that the stochastic steady state conditions depend on variance and covariance terms of consumption, productivity, and collateral premium. Because while $Var_t(C_{t+1})$ and $Cov_t(C_{t+1},A_{t+1})$ are positive, the sign of $Cov(C_{t+1},\eta_{t+1})$ depends on to the nature of the shock distribution, it is not immediately obvious what impact the second-order terms have on the steady-state level of capital (to see why the sign of $Cov_t(C_{t+1},A_{t+1})$ and $Cov(C_{t+1},\eta_{t+1})$ are positive, see Appendix 9.2.2). The impact of the second order terms depends on to the degree of risk aversion and precautionary saving behavior the household display. The gross-interest rate on the government bonds, just like the capital decision rule, also embodies second-order impacts as well.

The value of η is given by (19) and (20) with steady state conditions:

$$1+\eta = \int_0^{\varphi} \Phi(A_{t+1})dA_{t+1} + \int_{\varphi}^{\infty} \frac{A_{t+1}}{\varphi}\Phi(A_{t+1})dA_{t+1} \tag{56}$$

where $\varphi = \frac{(1-\delta)K+RB}{(1-\tau)(1-\alpha)K^{\alpha}}$. Note that the stochastic steady-state value of the collateral premium we use is the *expected* collateral premium, not the realized value. The government's budget constraint in the steady state becomes:

$$\tau AK^{\alpha} = (R-1)B. \tag{57}$$

Derivation of the steady state conditions can be found in Appendix B section 8.2.1., and inferences on $Var_t(C_{t+1})$, $Cov_t(C_{t+1},A_{t+1})$ and $Cov_t(C_{t+1},\eta_{t+1})$ can be found in Appendix B section 8.2.2..

Collateral Premium

This section includes numerically computed stochastic steady-state values given different level of government bond supplies.

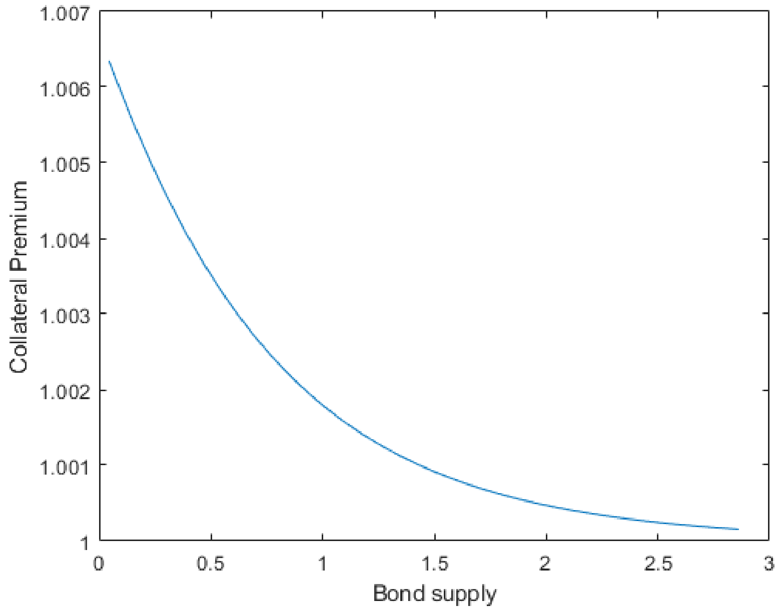


Figure above shows the steady state level of collateral premium $1+\eta$ at different levels of the steady state bond supplies. This is because higher steady state bond supply requires higher tax rate to finance the debt service. From the equation for the collateral premium, we can see how higher volume of bond supply, along with higher tax rate, reduces the collateral premium. From

$$\varphi = \frac{(1-\delta)K + RB}{(1-\tau)(1-\alpha)K^{\alpha}}, \tag{58}$$

rise in B and τ increases φ, which then changes the value of the collateral premium since

$$1+\eta = \int_0^{\varphi} \Phi(A_{t+1}) dA_{t+1} + \int_{\varphi}^{\infty} \frac{A_{t+1}}{\varphi} \Phi(A_{t+1}) dA_{t+1} \tag{59}$$

is decreasing in φ. It is intuitive and easy to see that the provision of larger volume of government bonds relax the collateral constraint, and thus reduce collateral premium. This holds true in the steady-state as well. What remains us to check is whether this reduction in the collateral premium is welfare improving, since it accompanies increase in the distorting tax.

Capital

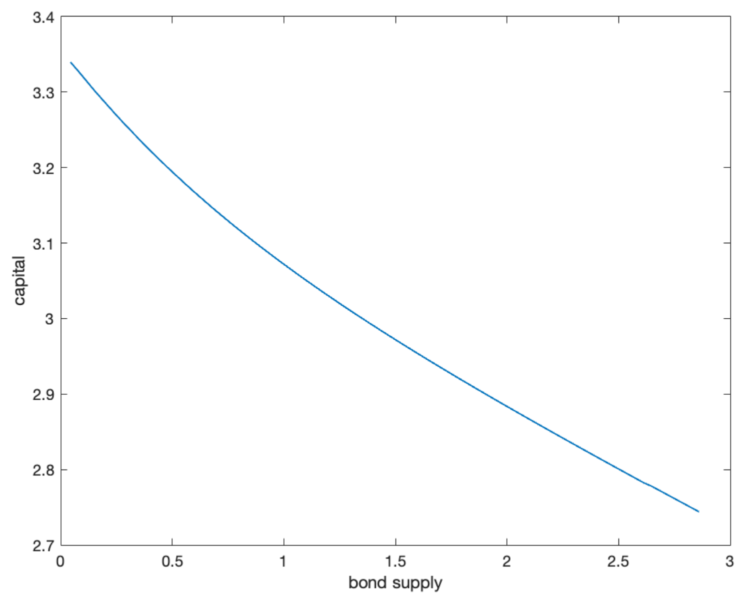


Figure above shows how capital is decreasing as the bond supply rises. Intuitively, there are two forces here that are contributing to the fall in the capital. First, the increase in the bond supply is accompanied by the increase in the tax rate, which distorts agent's capital choice. Secondly, the fall in the collateral premium relaxes the collateral requirement, reducing the capital volume. We can rewrite the stochastic steady state equilibrium condition (27):

$$K = \left(\frac{(1-\tau)\alpha(A + \frac{A}{2C}Var_t(C_{t+1}) - \frac{1}{C}Cov_t(C_{t+1}, A_{t+1}))}{\frac{1}{\beta} + \frac{(1-\delta}{C}Cov_t(C_{t+1}, \eta_{t+1}) - \frac{(1-\delta)(1+\eta)}{2C}Var_t(C_{t+1}) - (1-\delta)(1+\eta)}\right)^{\frac{1}{1-\alpha}}. \tag{60}$$

From (60), we can see that the numerator of the RHS of the equation is decreasing in τ, and the denominator is decreasing in η. Since τ is increasing in the bond supply and the collateral premium η is decreasing in the bond supply. To further investigate this, we can log-decompose the equation above:

$$\begin{aligned}(1-\alpha)ln(K) = {} & ln\Big((1-\tau)\alpha(A + \frac{A}{2C}Var_t(C_{t+1}) - \frac{1}{C}Cov_t(C_{t+1}, A_{t+1}))\Big) \\ & - ln\Big(\frac{1}{\beta} + \frac{(1-\delta)}{C}Cov_t(C_{t+1}, \eta_{t+1}) - (1+\eta)\big(\frac{(1-\delta)}{2C}Var_t(C_{t+1}) - (1-\delta)\big)\Big). \end{aligned} \tag{61}$$

The first term in the RHS is the distortionary effect of tax rate to the log-capital, and the second term is the effect of the collateral value to the capital level. We can see that the steady-state level of capital is indeed a decreasing function of the tax rate and an increasing function of the collateral premium. As the tax rate is an increasing function of the bond supply while the collateral premium is decreasing in the bond supply, and increase in the bond supply would crowd out the capital. Conventional models also feature crowding-out of capital from an increase in the bond supply, but in this model, the magnitude of the crowding out is amplified by the decrease in the collateral premium. Further, unlike conventional models, this model features overaccumulation of capital when the bond supply is low

Welfare

The variance inference regarding the stochastic steady-state welfare can be found in the Appendix B section 8.2.3.. As stated earlier, because this analysis does not take into the transitional periods from one steady state to another, this analysis does not provide the result on what is the optimal supply of government bond. However, it gives us a general understanding how the steady-state welfare moves as the result of committing to one level of bond supply for an extended time.

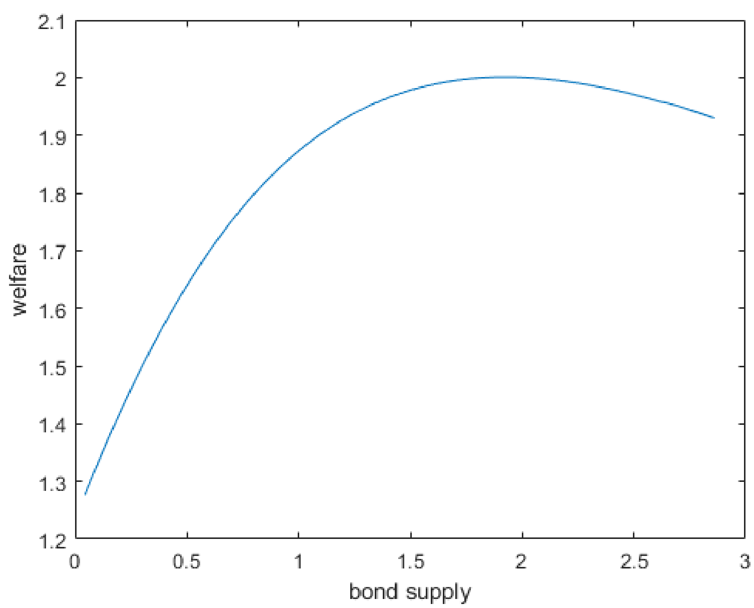
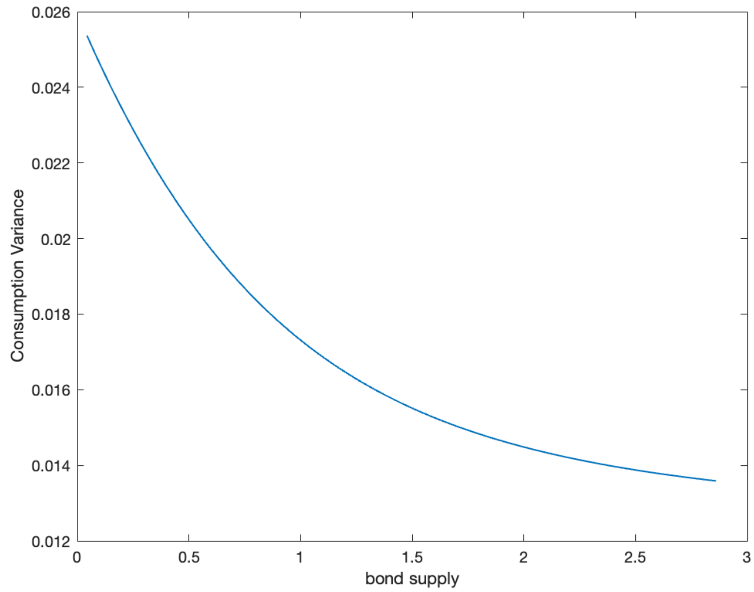


Figure above shows the second-order approximation of the expected lifetime welfare at the stochastic steady state. We see that there exists a bond supply that maximizes welfare around 1.8. The welfare increases in the bond supply as the consumption variance decreases. Below is the figure that displays the relationship of the bond supply and consumption variance.



This is because the increase in the bonds supply reduces the collateral premium, since the probability of collateral constraint binding decreases as the supply of government bond increases. This reduces the variance of collateral premium, leading to a fall in the consumption variance. Because the second order welfare approximation at the stochastic steady state takes in the consumption variance, the decrease in the consumption variance drives the initial increase in the expected utility. However, as the bond supply increases, capital is crowded out, and when the capital crowding out overwhelms the benefit from lowered variance, the expected welfare declines.

5.2 Comparing with the Lump-sum case

To understand how the distortion and the collateral premium affects agents' decisions, investigating the economy under lump-sum tax helps. In this scenario, we only observe the collateral relaxation effect of the bond supply without capital distortion.

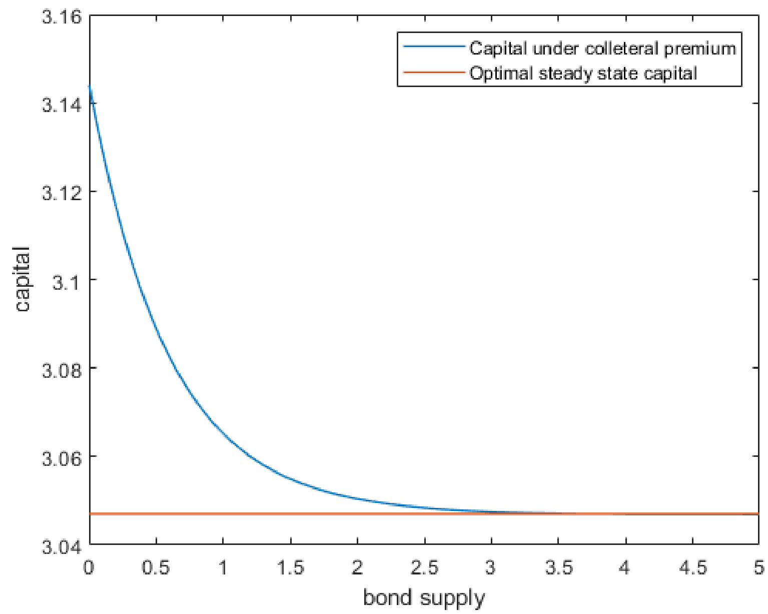
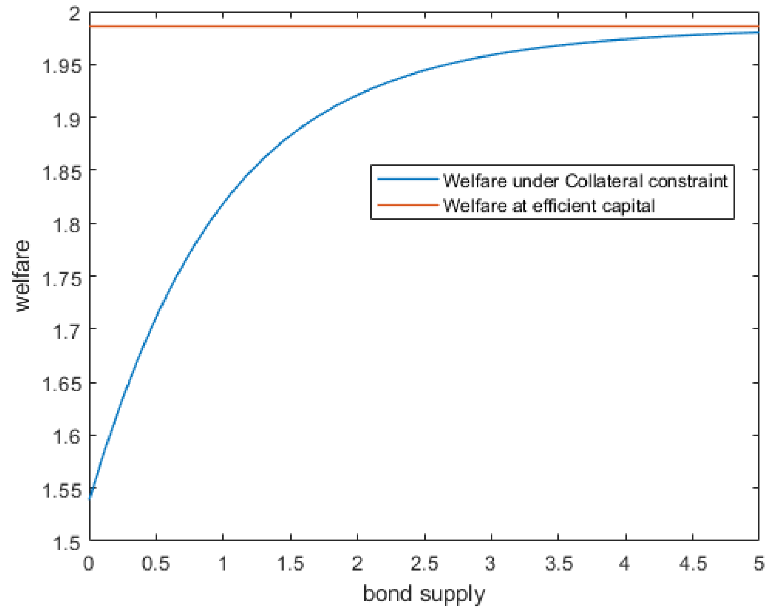
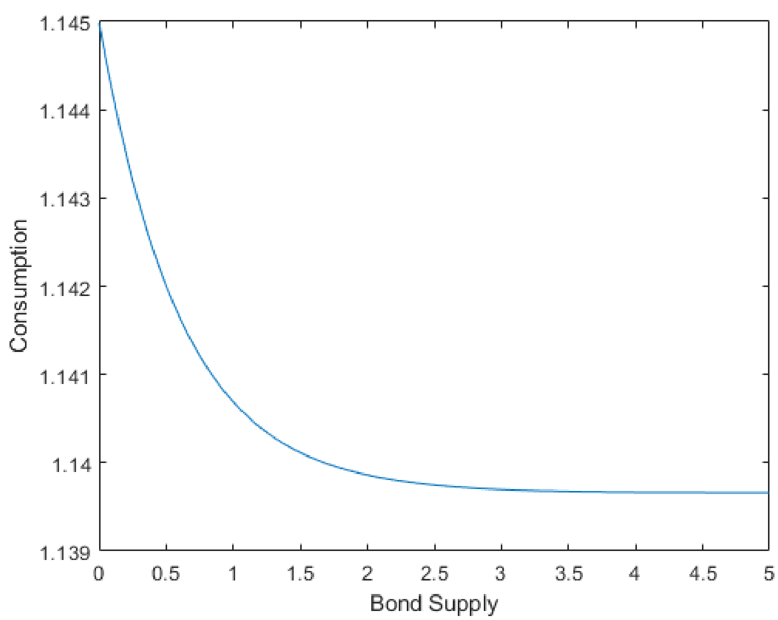
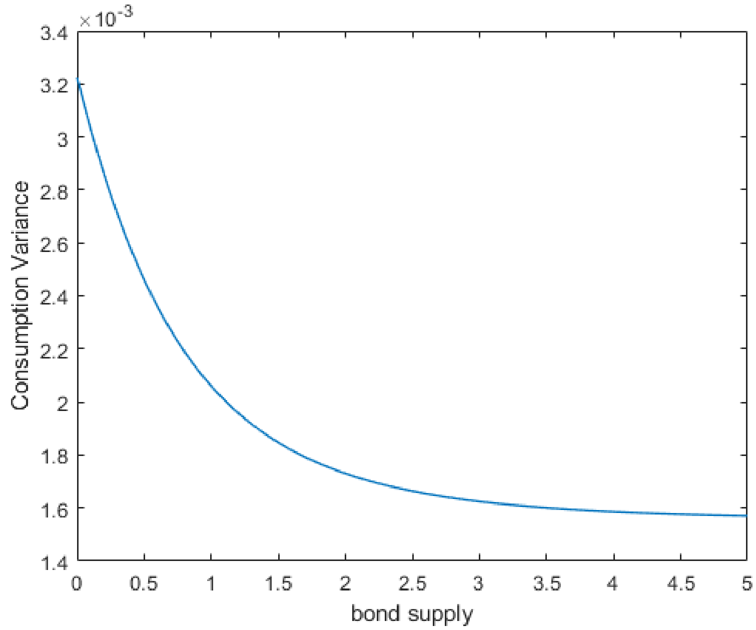


Figure above shows the steady state level of capital at different levels of bond supplies when the debt service is financed by lump-sum tax.The orange line shows the efficient level of the stochastic steady state capital. As shown, as the bond supply increases, the capital level asymptotically converges to the efficient steady state. The probability of collateral premium binding decreases, leading to a fall in the collateral premium. As the collateral premium diminishes, agents' need of capital for collateral purpose decreases. To achieve the efficient level of steady state, the collateral premium needs to diminish entirely. This suggests the optimal bond supply is infinite. This is more clear when we take a look at the expected lifetime utility at the steady state:



From the figure above, we can see the steady state welfare increases and asymptotically converges to the efficient level as the bond supply increases. The increase in the welfare, does not come from the increase in consumption, as in the case for the distorting tax. The increase in the welfare comes from the decrease in the consumption variance, as it shows evident in the following figures:





The steady state consumption level falls as the bond supply increases and capital level falls. The variance, as well as the consumption level, falls. The fall in the variance mostly comes from the fall in the collateral premium - the variance of collateral premium falls, feeding into the fall in the consumption variance as shown in the following figure. Note that consumption variance converges slower than the case with distorting tax. This is because in the lump-sum case, there is no real consequence of increasing the bond supply contrary to the distorting tax case. In the case for the distorting tax, on the contrary, the capital decreases not only due to the fall in the collateral premium, but also the increase in distorting tax.

6 Optimal Supply of Government Debt

In this section, I find the optimal supply of government debt assuming the economy is at a steady-state. I use projection method to approximate the policy function as a second-order Chebyshev polynomial of state variable. Contrary to the steady-state welfare discussion provided in the previous section, the welfare analysis discussed here does not compare steady-state values - it finds the bond supply that maximizes lifetime utility. Further, I show that the optimal bond supply increases as the standard deviation of the productivity shock increases.

The policy functions are given as functions of state variables, A_t and K_t, and the policy variable B_{t+1}:

$$K_{t+1} = a_0 + a_1K_t + a_2A_t + a_3B_{t+1} + a_4(2K_t^2 - 1) + a_5K_tA_t + a_6K_tB_{t+1} + a_7(2A_t^2 - 1) + a_8A_tB_{t+1} + a_9(2B_{t+1}^2 - 1) \quad (62)$$

$$Q_{t+1} = b_0 + b_1K_t + b_2A_t + b_3B_{t+1} + b_4(2K_t^2 - 1) + b_5K_tA_t + b_6K_tB_{t+1} + b_7(2A_t^2 - 1) + b_8A_tB_{t+1} + b_9(2B_{t+1}^2 - 1). \quad (63)$$

a_0 ... a_9 and b_0 ... b_9 are coefficients associated with the Chebyshev polynomial of capital and bond price respectively. I find these values numerically such that first-order-conditions, collateral constraints and resource constraints are satisfied in the variable grids. The policy functions suggests that agents in the economy makes decisions based on the realization of the productivity shock, the carried over capital volume, and the government's decision on the bond supply. The first two determines current period's production and wealth, while the last determines the future chance of collateral constraint binding and the tax rate. By mapping the expected lifetime utility at different bond supply, one can find the optimal supply of government debt B_{t+1}.

I assume that the initial state of the economy is in the steady-state with initial bond supply at zero: $B_t = 0$. I use log-utility function ((i.e. $\gamma = 1$). From the following figure, we can see how the economy reacts to an increase in the bond supply.

Figure1 introduces expected lifetime utility, capital investment decision, collateral premium, and the interest on government bonds at different bond supply B_{t+1} when the standard deviation of the underlying shock process is at $\sigma = 0.2$. Here, the optimal supply of government bonds is 0.4047, which amounts to 27.7 percent of the output. From the K_{t+1} plot, we can see that the government bonds crowd-out capital, as K_{t+1} is strictly decreasing in B_{t+1}. The collateral premium plots show that as the government increases the bond supply, the collateral premium decreases and converges to one. In this specific case, the collateral premium seems negligible, but yet it does not fully converge to one at the optimal supply of government bonds. R_{t+1}, the interest on government bonds shows a rather interesting behavior. First it rises as the bond supply increases, then it falls as

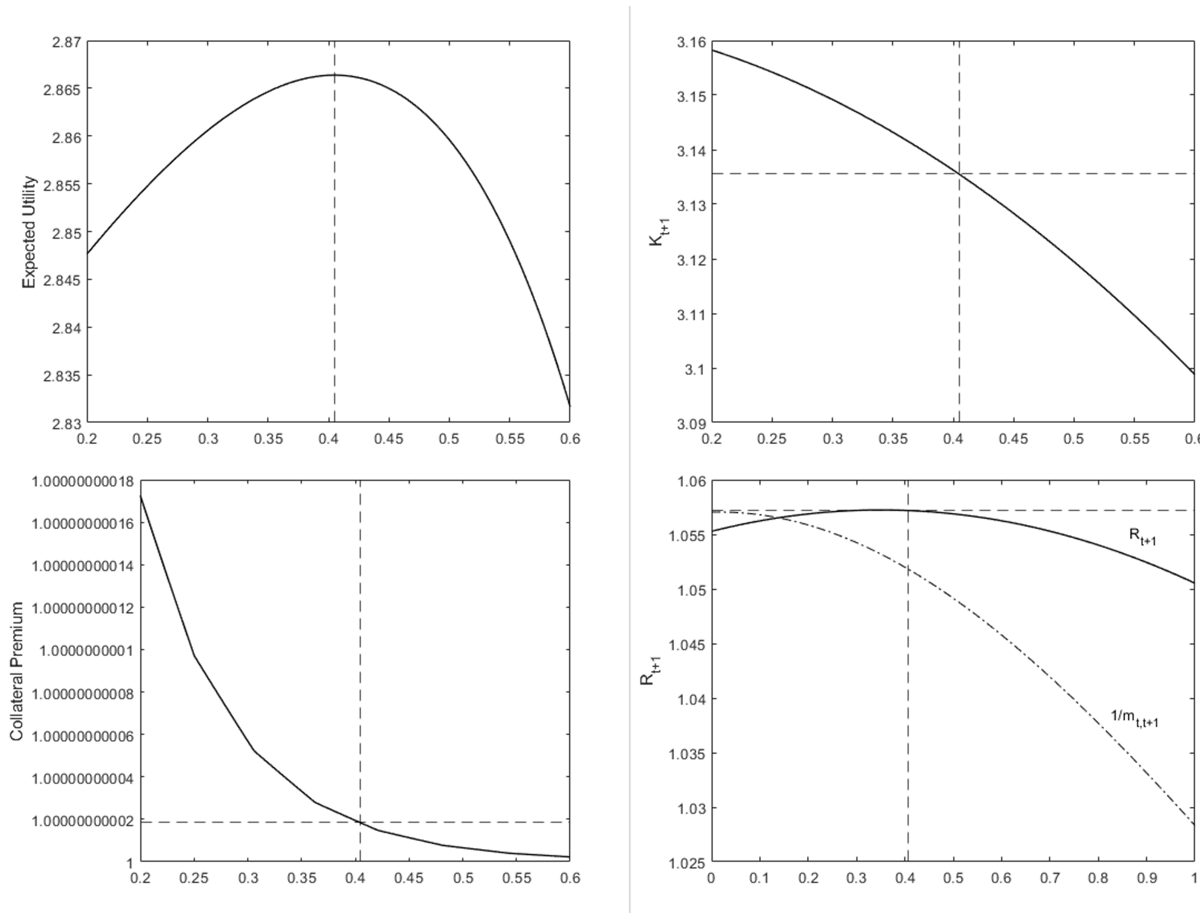


Figure 1: Expected lifetime utility on bond supply at different standard deviations, $\sigma = 0.2$

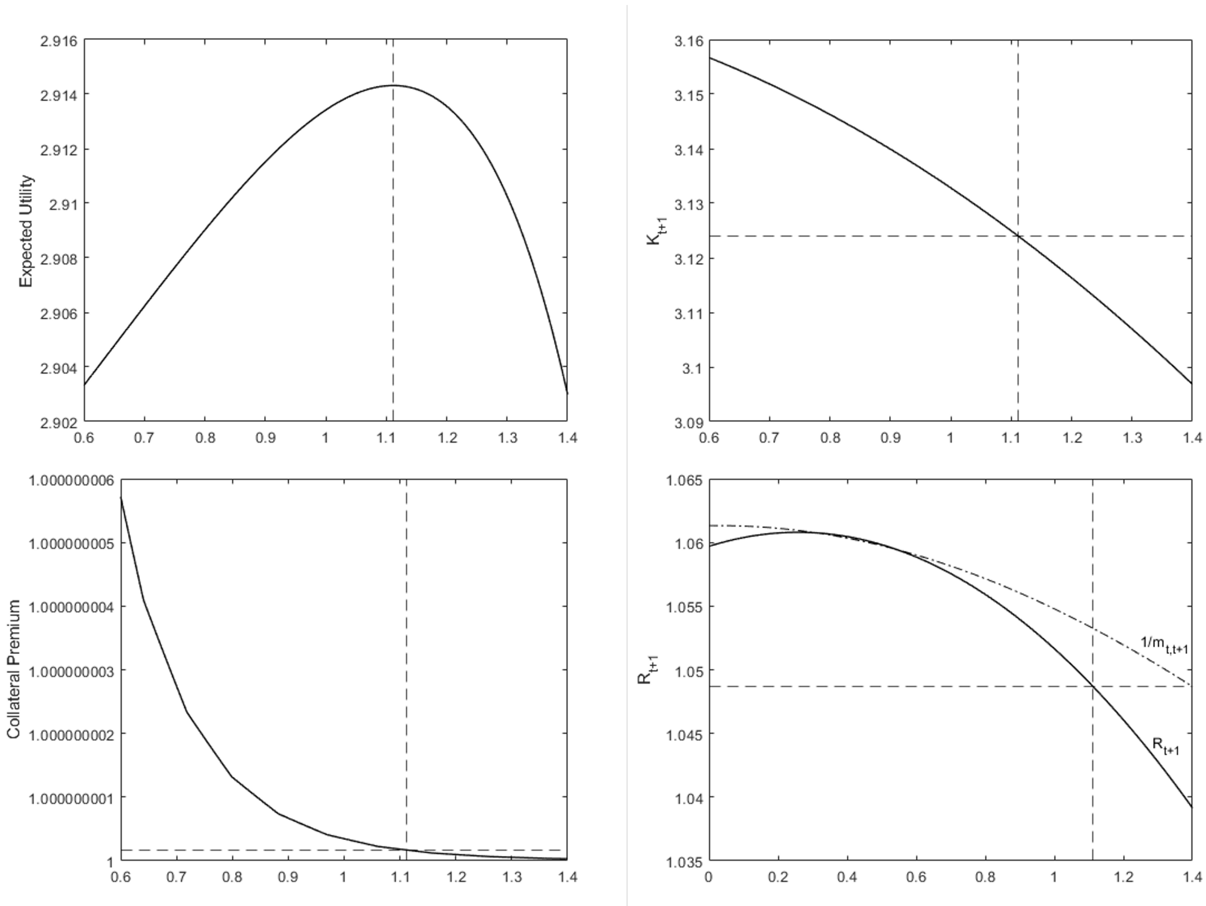


Figure 2: Expected lifetime utility on bond supply at different standard deviations, $\sigma = 0.25$

bond supply increases further. The initial rise of the government bonds is due to the relaxation of the collateral constraint - as the bond supply increases, the chance of collateral constraint binding in the following period decreases. One can alternatively describe this rise in the interest rate as bond price becoming 'cheaper.' However, because K_{t+1} is decreasing in B_{t+1}, as the bond supply increases, consumption today increases while the expected consumption at $t+1$ decreases - leading to a fall in the interest rate.

Figure 2 shows the same sets of plots, but with higher standard deviation at $\sigma = 0.25$. Here, we can see from the expected lifetime utility curve that the optimal supply of government bonds is larger than before at $B_{t+1} = 1.1123$ - this amounts to 76.08 percent of the output. At the optimal bond supply, because the supply is larger, it crowds-out larger volume of capital such that the capital investment is lower compared to the case with smaller standard deviation. One can also note that the interest rate on government bonds is lower compared to the previous case. This is because of lower capital investment decision. Collateral premium, on the other hand, is larger in this case. Because the level of uncertainty is larger with this example, their is a higher chance that the collateral constraint will bind in the future, which requires a much larger level of bond supply. Note that the probability of collateral constraint binding is not entirely gone at the optimal supply of government debt.

This result provides some valuable insights on our understanding of the role of government bonds in the economy. Because it serves as collateral, a low volume of bond supply can cause capital hoarding. As the production uncertainty increases, the chance of collateral constraint binding increases, which then increases the collateral premium. This exacerbates the capital hoarding. As the government increases the bond supply, it can reduce the capital hoarding that causes inefficiency. This effect becomes larger as the degree of the uncertainty increases. With larger production uncertainty, there is a higher probability of collateral constraint binding in the future. This increases the magnitude of the inefficiency, and the welfare maximizing bond supply increases.

We can interpret this result further by digging down on what makes the collateral premium higher. When the production uncertainty grows, the collateral premium increases together. This is because with higher uncertainty, there exists higher chance of the right tail of the distribution compared to the low uncertainty case. The collateral constraint would bind when the realized value of the productivity parameter A_{t+1} is high - with a mean preserving variance spread, the probability that $A_{t+1} > E_t(A_{t+1})$ becomes higher (so is the case with $A_{t+1} < E_t(A_{t+1})$, but this case does not affect the collateral premium). In other words, the firm awards higher collateral premium to capital and government bonds when the economy has higher potential to out-perform the expectation due to larger uncertainty. This could come as a somewhat counter intuitive behavior for a firm, as it needs to prepare for the case 'things work out better than expected.' When the firm feels higher upward pressure in terms of future productivity, it would accumulate higher capital to extract larger production out of it. However, the upward pressure to the productivity raises labor productivity as well, giving the collateral constraint larger pressure, driving up the collateral premium. So the effect of upward pressure to the productivity is amplified by the collateral premium, leading to an overaccumulation. To relax this, the government can increase the volume of bond supply. This not only reduces the collateral premium by relaxing the collateral constraint, but also raises the distorting tax, reducing the capital accumulation further.

7 Conclusion

The characterized results of the analytical example explicitly show us that the capital decision rule of the firm is a decreasing function of government bonds – which agrees with conventional results of literature. However, because in the absence of government bonds, agents tend to accumulate larger volume of capital than efficient level due to financial friction (or the collateral requirement), the government can supply bonds to optimally crowd out capital. This happens through two channels: by relaxing the collateral demand and by increasing the distortionary tax. In a more general environment, the computational result verifies the results from the characterized analytical example.

When the production uncertainty increases, agents increase their capital accumulation. In a general case, this would be because agents' demand for collateral increases and precautionary saving increases. However, in the analytical example, we focused on the increase in the capital accumulation due to the increase in collateral demand. Because the capital accumulation increases, the government needs to increase its bond supply to achieve the efficient level when there is an increase in production uncertainty. Through the stochastic steady-state analysis, we observed how bond supply tames consumption variance, which propagates through future uncertainty. This results in a meaningful welfare difference in the steady state.

Through the numerically solved policy functions and welfare analysis, I find the optimal supply of government bonds that maximizes welfare. Further, I show that the optimal bond supply increases as the magnitude of uncertainty grows. This is because higher uncertainty results in higher chance of collateral constraint binding in the future, leading to higher collateral premium on government bonds today. This incurs high utility cost, and to resolve this it requires larger volume of government bonds.

This work provides a framework for how we can think about fiscal policy's role in the financial market. As shown, the role of the fiscal policy in the financial market is significant as fiscal authorities' bond issuances affect the financial market by a huge magnitude. This is why central banks and fiscal policies of many countries get involved in controlling the market circulating volume of government bonds. At the same time, because private assets can provide similar liquidity services as collateral to a limited magnitude, we see a large volume of private substituting government bonds in the financial market. With this framework, we can understand how the market supply of government bonds affects agents' capital decision rule in a precise manner. For instance, when a policy decision maker sees an increase in the production uncertainty, this model can aid the understanding of the optimal market supply of government bonds need to increase.

8 References

Aiyagari, S. Rao and McGrattan, Ellen R. 1998. “The Optimum Quantity of Debt." *Journal of Monetary Economics,* 42:447-469.

Berentensen, Aleksander and Waller, Christopher. 2017. “Liquidity Premiums on Government Debt and the Fiscal Theory of the Price Level." *FRB St. Louis Working Paper No. 2017-8.*

Calvo, Gullermo, 1978. “On the Time Consistency of Optimal Policy in a Monetary Economy," Econometrica, Vol. 46 (6), pages. 1411-1428.

Chari, V.V. and Kehoe, Patrick J., 1999. “Optimal fiscal and monetary policy," Handbook of Macroeconomics, in: J. B. Taylor & M. Woodford (ed.), Handbook of Macroeconomics, Vol. 1, pages 1671-1745.

Coeurdacier, Nicolas, Helene Rey, Pablo Winant. 2011. “The Risky Steady State." *American Economic Review*, 101 (3): 398-401.

Diamond, Peter. 1965. “National debt in a neoclassical growth model". *American Economic Review*, 55 (5): 1126–1150.

Friedman, Milton, 1969. “The Optimum Quantity of Money: and Other Essays." *Aldine Publishing Company*. Chicago.

Greene, Andrew, Justyna Prodziemska, Robert Toomey. 2018. “Repo Market Fact Sheet." *Securities Industry and Financial Markets Association*.

Gorton, Gary, Stefan Lewellen, and Andrew Metrick. 2012. “The Safe-Asset Share." *American Economic Review*, 102 (3): 101-06.

Gorton, Gary and Ordonez, Guillermo. 2014. “Collateral Crises." *American Economic Review,* 104(2):343-78.

Gorton, Gary and Ordoñez, Guillermo, 2020. "Good Booms, Bad Booms," Journal of the European Economic Association, vol 18(2), pages 618-665.

Krishnamurthy, Arvind and Vissing-Jorgensen, Annette. 2012. “The Aggregate Demand for Treasury Debt." *Journal of Political Economy*, 102(2):233-67.

Schmitt-Grohe, Stephanie and Uribe, Martin. 2004. “Solving Dynamic General Equilibrium Models Using a Second-order Approximation to the Policy Function." *Journal of Economic Dynamics & Control*, 28: 755-775.

Sims, Christopher, 2022. “Optimal Fiscal and Monetary Policy with Distorting Taxes," Working paper, Princeton University.

Woodford, Michael, 1990 “The optimum quantity of money,” Handbook of Monetary Economics, Vol. 2, pages 1067–1152.

9 Appendix

9.1 Appendix A - Analytical Example

9.1.1 Conditions that ensure binding collateral constraint

For (41) to hold, we require following condition:

$$A^L_{t+1} \leq (1-\delta)K^*_{t+1} \tag{64}$$

which means that the collateral constraint will not bind if $A_{t+1} = A^L_{t+1}$. This condition can be elaborated to

$$(\frac{1}{\beta(1-\delta)-1-\rho})A^L_{t+1} \leq (1-\rho)A^H_{t+1}, \tag{65}$$

which can be ensured when we have sufficiently large ρ and A^H_{t+1}. (64) would suggest that the collateral constraint at $t+1$ will not bind when $A_{t+1} = A^L_{t+1}$ even when there is no government bonds supplied in the market. To ensure that there exists B^*_t that achieves the efficient capital allocation, we need a very large ρ - close to 1 - and A^H_{t+1}. We would not require such assumptions and requirements had the distribution been continuous in an infinite support. However, since we work with a binomial distribution for tractability, we rely on these assumptions that makes the analysis analogous to the mentioned case.

9.1.2 Capital decision as a function of bond supply

From (45), we have

$$K_{t+1} = \frac{\beta E_t(A_{t+1})}{1+\beta\frac{B_{t+1}}{K_{t+1}} - \beta(1-\delta)(\rho+(1-\rho)\frac{A^H_{t+1}}{(1-\delta)K_{t+1}+B_{t+1}})}. \tag{66}$$

By rearranging this optimal condition for K_{t+1}, we acquire a quadratic equation for K_{t+1} in terms of parameter values, $E_t(A_{t+1})$ and B_{t+1}. Solving for the quadratic equation, the positive solution is given by

$$K_{t+1} = \frac{1}{2(1-\delta)(1-\xi\rho)}\Bigg[\xi(E_t(A_{t+1})+(1-\rho)A^H_{t+1})-(1+\xi)B_{t+1}+$$
$$+\left((\xi(E_t(A_{t+1})+(1-\rho)A^H_{t+1})-(1+\xi)B_{t+1})^2+4(1-\delta)(1-\xi\rho)(\beta E_t(A_{t+1})+\xi\rho-\beta B_{t+1})B_{t+1}\right)^{\frac{1}{2}}\Bigg] \tag{67}$$

where $\xi \equiv \beta(1-\delta)$. Although the length of the equation might makes it harder for us to see what is going on, we can see that for a large enough B_{t+1}, it is ensured that K_{t+1} is decreasing in B_{t+1}. Further, K_{t+1} is decreasing in B_{t+1} if the condition below is satisfied:

$$E_t(A_{t+1})((1+\xi)\xi-2(1-\delta)(1-\xi\rho)\beta)+(1+\xi)\xi(1-\rho)A^H_{t+1} > 2\xi\rho(1-\delta)(1-\xi\rho) \tag{68}$$

which is a relation of the parameter values that can be ensured to hold given we have ρ sufficiently close to 1 and A^H_{t+1} large enough.

9.1.3 Output volatility and optimal supply of government bonds

From the binomial process of A_{t+1}, the variance of A_{t+1} is given by

$$\sigma^2 = (A^H_{t+1}-A^L_{t+1})^2\rho(1-\rho) \tag{69}$$

With some rearranging, this allows us to write the upperbound of the support A^H_{t+1} as

$$A^H_{t+1} = \frac{\rho\sigma}{\sqrt{\rho(1-\rho)}}+E_t(A_{t+1}), \tag{70}$$

where we assume $E_t(A_{t+1})$ and ρ are given, and A^H_{t+1} depends on the variance of the process. Substituting this identity into the capital decision rule (67), we can express capital as as function of the standard deviation σ.

$$K_{t+1} = \frac{1}{2(1-\delta)(1-\xi\rho)}\Bigg[\xi((2-\rho)E_t(A_{t+1})+\sigma\sqrt{\rho(1-\rho)})-(1+\xi)B_{t+1}+ \\ +\Bigg(\xi((2-\rho)E_t(A_{t+1})+\sqrt{\rho(1-\rho)}\sigma)-(1+\xi)B_{t+1})^2+ \\ +4(1-\delta)(1-\xi\rho)(\beta E_t(A_{t+1})+\xi\rho-\beta B_{t+1})B_{t+1}\Bigg)^{\frac{1}{2}}\Bigg] \quad (71)$$

and the capital stock at the absence of government bonds $\bar{K}_{t+1}$ can be written as a function of the standard deviation σ:

$$\bar{K}_{t+1} = \frac{\beta((2-\rho)E_t(A_{t+1})+\sigma\sqrt{\rho(1-\rho)})}{1-\beta(1-\delta)\rho}. \quad (72)$$

Now consider a mean preserving spread of the process for A_{t+1}: the value for $E_t(A_{t+1})$ remains the same, while σ increases. This would be the type of spread where ρ remains the same, while the spread $A^H_{t+1}-A^L_{t+1}$ increases. Because K_{t+1} and $\bar{K}_{t+1}$ are increasing in σ, we see that the agents will accumulate larger volume of capital with the increase in the variance. The optimal supply of government bonds B^*_1 becomes

$$B^*_t = \frac{1}{2\beta}\Bigg[(\beta E_t(A_{t+1})+\xi\rho)-K^*_{t+1}(1+\xi)+\Bigg((K^*_{t+1}(1+\xi)-(\beta E_t(A_{t+1})+\xi\rho))^2+ \\ +4\beta(\xi((2-\rho)E_t(A_{t+1})+\sigma\sqrt{\rho(1-\rho)})-(1-\delta)(1-\xi\rho)K^*_{t+1})K^*_{t+1}\Bigg)^{\frac{1}{2}}\Bigg] \quad (73)$$

where we also see that the optimal supply of government bonds would increase when the variance of the output productivity increases.

9.2 Appendix B - Stochastic steady-state

9.2.1 Second-order expansion of the Euler equation

Assume their exists a stochastic steady state of the economy. To characterize the stochastic steady state, we expand the first order conditions to the second order. Rearranging first order condition (13), we get

$$E_t\Big[\beta\frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}}\Big((1-\tau_{t+1})A_{t+1}\alpha K_{t+1}^{\alpha-1}+(1-\delta)(1+\eta_{t+1})\Big)\Big] = 1. \quad (74)$$

For our analysis, we assume the same policy rule is applied as the government chooses the bond supply B_{t+1}, where they aim to supply the same amount of bond in the future: $E_t(B_{t+2}) = B_{t+1}$. This nails down τ_{t+1} at period t. Current tax rate τ_t is then given by the interest payment made at current period and the government's choice of B_{t+1}. This provides us with the following condition for the bond supply today and the following period's tax rate:

$$\tau_{t+1} E_t(A_{t+1}) K_{t+1}^{\alpha} = (R_{t+1} - 1) B_{t+1}. \tag{75}$$

With this assumption, the first order Taylor expansion of the equation with respect to C_{t+1}, A_{t+1}, and η_{t+1} gives us following:

$$\begin{aligned}
&\beta \frac{E_t(C_{t+1})^{-\gamma}}{C_t^{-\gamma}} \Big((1-\tau_{t+1}) A \alpha K_{t+1}^{\alpha-1} + (1-\delta) E_t(1+\eta_{t+1}) \Big) \\
&\quad - \beta\gamma \frac{E_t(C_{t+1})^{-\gamma-1}}{C_t^{-\gamma}} \Big((1-\tau_{t+1}) A \alpha K_{t+1}^{\alpha-1} + (1-\delta) E_t(1+\eta_{t+1}) \Big) E_t \Big(C_{t+1} - E_t(C_{t+1}) \Big) \\
&+ \beta \frac{E_t(C_{t+1})^{-\gamma}}{C_t^{-\gamma}} (1-\tau_{t+1}) \alpha K_{t+1}^{\alpha-1} E_t(A_{t+1} - A) + \beta \frac{E_t(C_{t+1})^{-\gamma}}{C_t^{-\gamma}} (1-\delta) E_t \Big(\eta_{t+1} - E_t(\eta_{t+1}) \Big) = 1
\end{aligned} \tag{76}$$

where $A \equiv E_t(A_{t+1})$.

The second order expansion is then given as following:

$$
\begin{aligned}
&\beta\frac{E_t(C_{t+1})^{-\gamma}}{C_t^{-\gamma}}\Big((1-\tau_{t+1})A\alpha K_{t+1}^{\alpha-1}+(1-\delta)E_t(1+\eta_{t+1})\Big)\\
&\quad-\beta\gamma\frac{E_t(C_{t+1})^{-\gamma-1}}{C_t^{-\gamma}}\Big((1-\tau_{t+1})A\alpha K_{t+1}^{\alpha-1}+(1-\delta)E_t(1+\eta_{t+1})\Big)E_t\Big(C_{t+1}-E_t(C_{t+1})\Big)\\
&\quad+\beta\frac{E_t(C_{t+1})^{-\gamma}}{C_t^{-\gamma}}(1-\tau_{t+1})\alpha K_{t+1}^{\alpha-1}E_t\Big(A_{t+1}-A\Big)+\beta\frac{E_t(C_{t+1})^{-\gamma}}{C_t^{-\gamma}}(1-\delta)E_t\Big(\eta_{t+1}-E_t(\eta_{t+1})\Big)\\
&+\frac{1}{2}\beta\gamma(\gamma+1)\frac{E_t(C_{t+1})^{-\gamma-2}}{C_{t+1}^{-\gamma}}\Big((1-\tau_{t+1})A\alpha K_{t+1}^{\alpha-1}+(1-\delta)E_t(1+\eta_{t+1})\Big)E_t\Big[\Big(C_{t+1}-E_t(C_{t+1})\Big)^2\Big]\\
&\quad-\beta\gamma\frac{E_t(C_{t+1})^{-\gamma-1}}{C_t^{-\gamma}}E_t(1-\tau_{t+1})\alpha K_{t+1}^{\alpha-1}E_t\Big[\Big(C_{t+1}-E_t(C_{t+1})\Big)\Big(A_{t+1}-A\Big)\Big]\\
&\quad-\beta\gamma\frac{E_t(C_{t+1})^{-\gamma-1}}{C_t^{-\gamma}}(1-\delta)E_t\Big[\Big(C_{t+1}-E_t(C_{t+1})\Big)\Big(\eta_{t+1}-E_t(\eta_{t+1})\Big)\Big]=1. \qquad (77)
\end{aligned}
$$

From the first order condition (15), we have

$$
\frac{1}{R_{t+1}}=E_t\Big(\beta\frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}}(1+\eta_{t+1})\Big) \qquad (78)
$$

which expands in the second order to

$$
\begin{aligned}
\frac{1}{R_{t+1}}&=\beta\frac{E_t(C_{t+1})^{-\gamma}}{C_t^{-\gamma}}E_t(1+\eta_{t+1})-\beta\gamma\frac{E_t(C_{t+1})^{-\gamma-1}}{C_t^{-\gamma}}E_t(1+\eta_{t+1})E_t\big(C_{t+1}-E_t(C_{t+1})\big)\\
&+\beta\frac{E_t(C_{t+1})^{-\gamma}}{C_t^{-\gamma}}E_t\big(\eta_{t+1}-E_t(\eta_{t+1})\big)+\frac{1}{2}\beta\gamma(\gamma+1)\frac{E_t(C_{t+1})^{-\gamma-2}}{C_t^{-\gamma}}E_t(1+\eta_{t+1})E_t[\big(C_{t+1}-E_t(C_{t+1})\big)^2]\\
&--\beta\gamma\frac{E_t(C_{t+1})^{-\gamma-1}}{C_t^{-\gamma}}E_t[\big(C_{t+1}-E_t(C_{t+1})\big)\big(\eta_{t+1}-E_t(\eta_{t+1})\big)] \qquad (79)
\end{aligned}
$$

9.2.2 Steady-state conditions

To nail down the values for $Var_t(C_{t+1})$, $Cov_t(C_{t+1},A_{t+1})$ and $Cov_t(C_{t+1},\eta_{t+1})$, we postulate a linear solution to the C_{t+1} in terms of A_{t+1} and η_{t+1}:

$$
C_{t+1}=C+\lambda_0+\lambda_1(A_{t+1}-A)+\lambda_2(\eta_{t+1}-\eta). \qquad (80)
$$

This allows us to write $Var_t(C_{t+1})$, $Cov_t(C_{t+1}, A_{t+1})$ and $Cov_t(C_{t+1}, \eta_{t+1})$ as follows:

$$Var_t(C_{t+1}) = \lambda_1^2 Var(A) + 2\lambda_1\lambda_2 Cov_t(A_{t+1}, \eta_{t+1}) + \lambda_2^2 Var_t(\eta_{t+1}) \tag{81}$$

$$Cov_t(C_{t+1}, A_{t+1}) = \lambda_1 Var(A) + \lambda_2 Cov_t(A_{t+1}, \eta_{t+1}) \tag{82}$$

and

$$Cov_t(C_{t+1}, \eta_{t+1}) = \lambda_1 Cov_t(A_{t+1}, \eta_{t+1}) + \lambda_2 Var_t(\eta_{t+1}), \tag{83}$$

where

$$Var_t(\eta_{t+1}) = E_t((1+\eta_{t+1})^2)) - (1+\eta)^2 = \int_0^{\varphi} \Phi(A_{t+1})dA_{t+1} + \int_{\varphi}^{\infty} (\frac{A_{t+1}}{\varphi})^2 \Phi(A_{t+1})dA_{t+1} - (1+\eta)^2 \tag{84}$$

and

$$Cov_t(A_{t+1}, \eta_{t+1}) = E_t(A_{t+1}\eta_{t+1}) - A\eta = \int_0^{\varphi} A_{t+1}\Phi(A_{t+1})dA_{t+1} + \int_{\varphi}^{\infty} \frac{A_{t+1}^2}{\varphi}\Phi(A_{t+1})dA_{t+1} - A\eta. \tag{85}$$

λ_0, λ_1, and λ_2 are given from the first order expansion of the first order condition (76):

$$\begin{aligned} &C_{t+1} - C = \\ &\frac{C}{\gamma} + \frac{C}{\beta\gamma\big((1-\tau)A\alpha K^{\alpha-1} + (1-\delta)(1+\eta)\big)}\big(\beta(1-\tau)\alpha K^{\alpha-1}(A_{t+1}-A) + \beta(1-\delta)(\eta_{t+1}-\eta) - 1\big) \end{aligned} \tag{86}$$

which yields

$$\lambda_0 = \frac{C}{\gamma} + \frac{C}{\beta\gamma\big((1-\tau)A\alpha K^{\alpha-1} + (1-\delta)(1+\eta)\big)} \tag{87}$$

$$\lambda_1 = \frac{C(1-\tau)\alpha K^{\alpha-1}}{\gamma\big((1-\tau)A\alpha K^{\alpha-1} + (1-\delta)(1+\eta)\big)} \tag{88}$$

and

$$\lambda_2 = \frac{C(1-\delta)}{\gamma\big((1-\tau)A\alpha K^{\alpha-1} + (1-\delta)(1+\eta)\big)}. \tag{89}$$

Put together, the stochastic steady state is given by the values of C, K, R, B, and η that solves the system of equations (53), (54),(55),(56), and (57), given $\tau \in [0, 1)$.

9.3 Steady State Welfare

9.3.1 Approximating expected utility around the steady state

Although the economy is expected remain in the same state in the steady state, the economy is still prone to the productivity shock. The expected life time utility is given by

$$E_t(\sum_{i=0}^{\infty}\beta^i \frac{C_{t+i}^{1-\gamma}}{1-\gamma}) \tag{90}$$

where $E_t(C_{t+i}) = C\ \forall i$. Expanding this to the second order and applying the steady state conditions, we get

$$\frac{1}{1-\beta}\frac{C^{1-\gamma}}{1-\gamma} - \sum_{i=1}^{\infty}\beta^i \frac{\gamma}{2} Var_t(C_{t+i}) \tag{91}$$

9.3.2 Variance inferences

As shown in the previous part, the value of $Var_t(C_{t+1})$ is given as follows:

$$Var_t(C_{t+1}) = \lambda_1^2 Var(A) + 2\lambda_1\lambda 2Cov_t(A_{t+1}, \eta_{t+1}) + \lambda_2^2 Var_t(\eta_{t+1}). \tag{92}$$

$Var_t(C_{t+i})$ for $i > 0$, on the other hand, needs some more inference. Because future choice of capital is not known in the current period, we use the linear postulation for C_{t+i} in terms of A_{t+i}, η_{t+i} and K_{t+i}:

$$C_{t+i} = C + \lambda_0 + \lambda_1(A_{t+i} - A) + \lambda_2(\eta_{t+i} - \eta) + \lambda_3(K_{t+i} - K). \tag{93}$$

From the linear expansion of the equilibrium condition (13), we get the value for λ_3 as

$$\lambda_3 = -\frac{C(1-\tau)A\alpha(1-\alpha)K^{\alpha-2}}{\gamma((1-\tau)A\alpha K^{\alpha-1} + (1-d)(1+\eta))} \tag{94}$$

and the values for λ_0, λ_1 and λ_2 remain as the same given from the previous part. This allows us to write

$$Var_t(C_{t+i}) = \lambda_1^2 Var(A) + \lambda_2^2 Var_t(\eta_{t+i}) + \lambda_3^2 Var_t(K_{t+i}) + 2\lambda_1\lambda_2 Cov_t(A_{t+i}, \eta_{t+i}) \\ + 2\lambda_2\lambda_3 Cov_t(\eta_{t+i}, K_{t+i}). \quad (95)$$

From the market clearing condition, we can write K_{t+i} as a function of C_{t+i-1}, A_{t+i-1}, and K_{t+i-1}:

$$E_t(K_{t+i}) = E_t(A_{t+i-1}K_{t+i-1}^{\alpha} + (1-\delta)K_{t+i-1} - C_{t+i-1}), \quad (96)$$

where the first order expansion of the market clearing condition gives us

$$E_t(K_{t+i}) = AK^{\alpha} + (1-\delta)K - C + K^{\alpha}(E_t(A_{t+i-1}) - A) \\ + A\alpha K^{\alpha-1}E_t(K_{t+i-1} - K) + (1-\delta)E_t(K_{t+i-1} - K) - E_t(C_{t+i-1} - C). \quad (97)$$

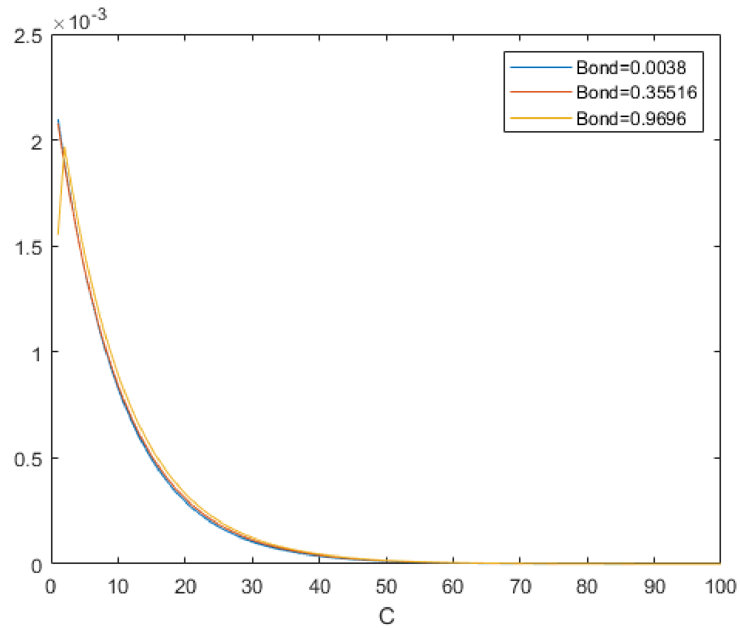
This allows us to write $Var_t(K_{t+i-1})$ as

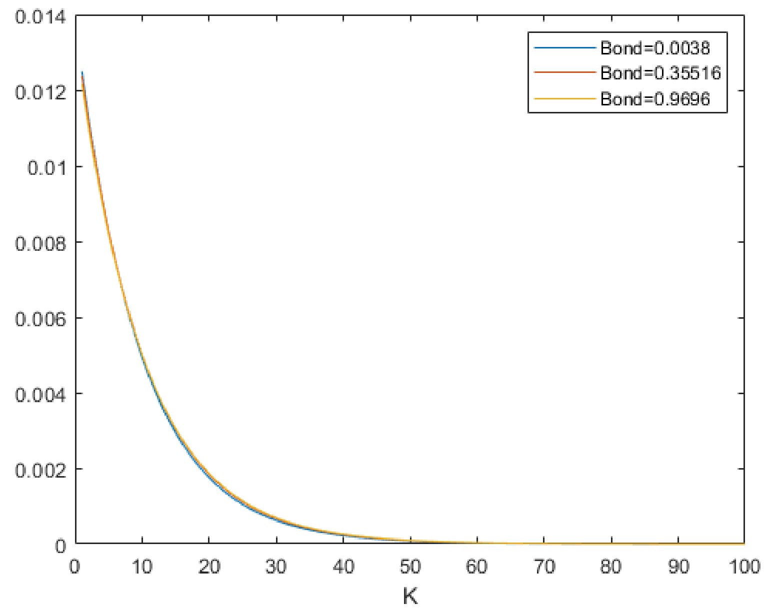
$$Var_t(K_{t+i}) = K^{2\alpha}Var(A) + (A\alpha K^{\alpha-1} + (1-\delta))^2 Var_t(K_{t+i-1}) + Var_t(C_{t+1-i}) - K^{\alpha}Cov_t(C_{t+i-1}, A_{t+i-1}). \quad (98)$$

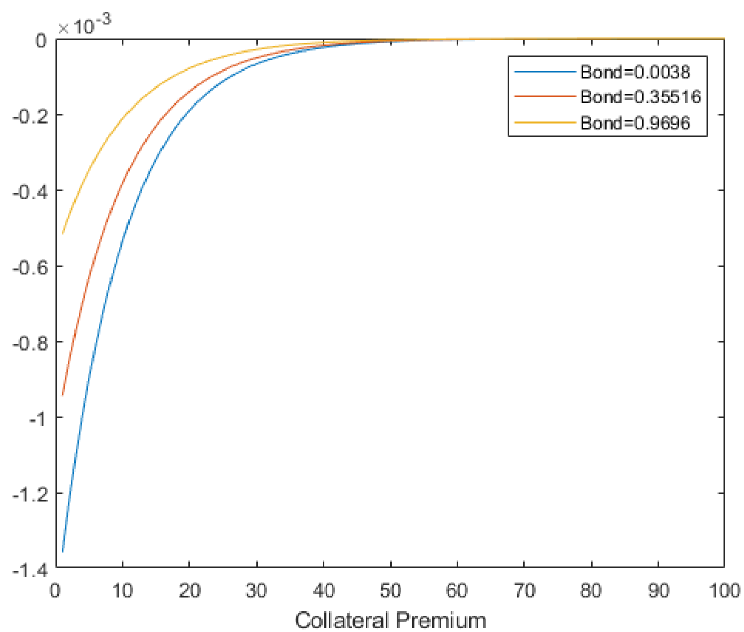
Note that the consumption variance and capital variance feeds into the variances of the future terms, thereby amplifying second-order effects. This allows us to capture the risk averse behavior of the household, as risk feeds into the welfare function. Further, although the first-order effect of capital crowd-out reduces steady-state consumption, it can be welfare improving if such crowd-out results in lower consumption variance: it feeds into future capital variance, amplifying its effect, eventually overpowering the first-order reduction in the consumption level. This mechanism allows one to capture how the risk-averse behavior manifests into decision rule. Moreover, it allows us to understand how the increase in the bond supply can be welfare improving as it reduces future consumption risks.

9.3.3 Impulse Response Functions

In this section, I present impulse response functions at three different levels of bond supply. I used the method provided by Schmitt-Grohe and Uribe to calculate the policy functions around the stochastic steady state. Impulse response functions are in linear terms, and do not carry second order dynamics. Note that each bond supply represented has its own steady state equilibrium. For each case, I equally give the shock of +0.01 to the productivity, A_t. From the initial shock of $A_t - A = 0.01$, we have following impulse responses:







Upon a positive productivity shock, we can observe that consumption and capital accumulation increases temporarily, and converges back to the steady state. The collateral premium reduces, as the capital stock increases due to the shock, the need for collateral drops. As the capital level converges to the steady state, the collateral premium converges to the steady state. Note that for consumption and capital deviation, there is no meaningful difference across different levels of bond supplies. However, the collateral premium shows somewhat different responses to the same magnitude of the shock depending on the level of bond supplies. We observe that the collateral premium is less prone to the shock when the bond supply is larger. This feeds into the welfare effect of the bond supply - as the collateral premium becomes less responsive/volatile to shocks, it has positive second order effect to the welfare. In other words, larger bond supply reduces variance of the collateral premium, hence increases welfare.